Vectors and Trigonometry

William F. Barnes

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1 Introduction to Vectors

A *vector* is an ordered list of numbers or variables condensed into one mathematical object. For instance, the sequence of numbers 2, 5, and 7 can be written $\vec{V} = \langle 2, 5, 7\rangle$, where V is the label for the vector, specially marked by an arrow ($\vec{}$). (Textbooks often denote vectors using bold letters instead of arrows. This is merely a notational convention.)

The individual members (or elements) of a vector are called *components*, and the total number of components is called the vector's *dimension*. The order in which the components of a vector are listed *does* matter. For example, the three-dimensional vector $\vec{V} = \langle 2, 5, 7\rangle$ is not equivalent to the reversed version $\langle 7, 5, 2\rangle$.

Any individual component is represented by the vector's symbol without the arrow, but including an index subscript. For instance, we could represent $\vec{V}$ as $\vec{V} = \langle V_a, V_b, V_c\rangle$ with $V_a = 2$, $V_b = 5$, $V_c = 7$, but the letters a, b, c could easily have been x, y, z, or perhaps i, j, k. Vector component labels are, after the dust settles, purely for bookkeeping.

1.1 Vector Addition and Subtraction

Two vectors of equal dimension can be combined to form a new vector of the same dimension by adding (or subtracting) equivalently-ranked components. Consider two vectors $\vec{X} = \langle 3, 6, 4\rangle$ and $\vec{Y} = \langle 6, -3, 2.2\rangle$. If we let $\vec{S}$ denote the sum of the two vectors, and likewise let $\vec{D}$ denote the difference, it follows that:

$$\vec{S} = \vec{X} + \vec{Y} = \langle 3 + 6, 6 + (-3)\,, 4 + 2.2\rangle = \langle 9, 3, 6.2\rangle$$

$$\vec{D} = \vec{X} - \vec{Y} = \langle 3 - 6, 6 - (-3)\,, 4 - 2.2\rangle = \langle -3, 9, 1.8\rangle$$

1.2 Multiplication by a Number

To multiply a vector by a real or complex number, more generally known as a *scalar*, multiply each component of the vector by that scalar. For example, multiplying $\vec{X} = \langle 3, 6, 4\rangle$ by a factor of 3 and calling the resultant vector $\vec{Z}$, we have

$$\vec{Z} = 3\vec{X} = 3\,\langle 3, 6, 4\rangle = \langle 9, 18, 12\rangle \ .$$

Obviously, the resultant vector $\vec{Z}$ has the same dimension as the original $\vec{X}$. Since $\vec{X}$ and $\vec{Z}$ only differ by a scale factor, the vectors somehow align or overlap.

Corollary

When any two vectors are identical up to a scale factor, they are said to be *parallel*.

1.3 Zero Vector

When we write a vector such as $\vec{V} = \langle 3, 5\rangle$, it's already been assumed (correctly) that the numbers 3 and 5 are each 'measured' (or referenced) from the number 0. Formally, the vector $\langle 3, 5\rangle$ is measured from the vector $\langle 0, 0\rangle$, known as the *zero vector*, which is trivial in the sense that all of its components are zero. The zero vector is defined in any number of dimensions, and is always denoted $\vec{0}$ (a zero with an arrow over it):

$$\vec{0} = \langle 0, 0, \ldots, 0 \rangle$$

1.4 Additive Inverse

The *additive inverse* of any vector $\vec{V}$ is written as $-\vec{V}$, and is defined such that

$$\vec{V} + (-\vec{V}) = \vec{0} \, .$$

In other words, a vector's additive inverse is some new vector that, when added to the original, yields the zero vector. This is a very long-winded way of saying that the additive inverse of a vector is simply the original vector with a minus sign in front of each component.

1.5 Algebraic Rules

Concerning addition, subtraction, and multiplication by a scalar, vector algebra conveniently behaves much like ordinary algebra, which means the usual rules of commutativity and associativity hold. Let $\vec{P}$, $\vec{Q}$, and $\vec{R}$ be three vectors of equal dimension, and let a and b be different scalars (real or complex). The following identities are always true (some appear completely trivial):

$\vec{P} + \vec{Q} = \vec{Q} + \vec{P}$	(commutativity)
$(\vec{P} + \vec{Q}) + \vec{R} = \vec{P} + (\vec{Q} + \vec{R})$	(associativity)
$\vec{P} + \vec{0} = \vec{P}$	(existence of zero vector)
$\vec{P} + (-\vec{P}) = \vec{0}$	(existence of additive inverse)
$a(\vec{P} + \vec{Q}) = a\vec{P} + a\vec{Q}$	(distributive property I)
$(a + b)\vec{R} = a\vec{R} + b\vec{R}$	(distributive property II)
$a(b\vec{P}) = (ab)\,\vec{P}$	(associativity with scalars)

1.6 Dot Product

The *dot product* or *scalar product* is a multiplication operation that tells us how much of one vector's 'shadow' falls upon another vector. The result is a scalar, sometimes called a *projection*. For two-dimensional vectors $\vec{A}$ and $\vec{B}$, the projection of $\vec{A}$ onto $\vec{B}$ is written as $\vec{A} \cdot \vec{B}$, is defined as

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y \, ,$$

and readily generalizes for N-dimensional vectors to be

$$\vec{A} \cdot \vec{B} = \sum_{k=1}^{N} A_k B_k \, .$$

By symmetry, the dot product is commutative

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \, ,$$

and obeys the distribution property

$$\vec{A} \cdot \left(\vec{B} + \vec{C}\right) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}\,.$$

Multiplication by the zero vector always yields zero:

$$\vec{A} \cdot \vec{0} = 0$$

Corollary

Vectors $\vec{A}$ and $\vec{B}$ are *perpendicular*, also known as *orthogonal*, if they have zero dot product:

$$\vec{A} \cdot \vec{B} = 0$$

1.7 Magnitude

The *magnitude* of a vector $\vec{V}$ with N components, denoted $\left|\vec{V}\right|$, or just plain V, is the square root of the dot product with itself:

$$\left|\vec{V}\right| = V = \sqrt{\vec{V} \cdot \vec{V}} = \sqrt{\sum_{k=1}^{N} V_k^2}$$

Expressed (perhaps) more transparently, the magnitude is like the 'hypotenuse' in a generalized version of the Pythagorean theorem:

$$V = \sqrt{V_1^2 + V_2^2 + V_3^2 + \cdots + V_N^2}$$

1.8 Cross Product

The *cross product* is a multiplication operation that mixes the components of two vectors, written $\vec{A} \times \vec{B}$, pronounced 'A cross B'. The cross product is antisymmetric when swapping the order of vectors:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Two Dimensions

For two-dimensional vectors $\vec{A}$ and $\vec{B}$, the cross product of $\vec{A}$ into $\vec{B}$ is written as $\vec{A} \times \vec{B}$, whose magnitude is

$$\left|\vec{A} \times \vec{B}\right| = A_x B_y - A_y B_x\,.$$

The direction that $\vec{A} \times \vec{B}$ points is perpendicular to both $\vec{A}$ and $\vec{B}$. That is, if $\vec{A}$ and $\vec{B}$ are embedded on your screen (and not parallel), the result $\vec{A} \times \vec{B}$ points perpendicular to the screen.

Three Dimensions

For three-dimensional vectors $\vec{A} = \langle A_x, A_y, A_z \rangle$ and $\vec{B} = \langle B_x, B_y, B_z \rangle$, the cross product $\vec{A} \times \vec{B}$ is given by

$$\vec{A} \times \vec{B} = \langle A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x \rangle \ ,$$

and is surely perpendicular to each of $\vec{A}$ and $\vec{B}$.

Corollary

The cross product of parallel vectors is the zero vector. If $\vec{A} = a\vec{B}$, then:

$$\vec{A} \times \vec{B} = a\left(\vec{B} \times \vec{B}\right) = \vec{0}$$

BAC-CAB Identity

A useful equation known as the *BAC-CAB identity*, reads

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right) \ .$$

To prove this, we use brute force (via optional determinant notation to contain the cross product):

$$\begin{aligned}
\vec{A} \times \left(\vec{B} \times \vec{C}\right) &= \vec{A} \times \left(\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}\right) \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix} \\
&= \langle A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z, 0, 0 \rangle + \\
&\quad \langle 0, A_z B_y C_z - A_z B_z C_y - A_x B_x C_y + A_x B_y C_x, 0 \rangle + \\
&\quad \langle 0, 0, A_x B_z C_x - A_x B_x C_z - A_y B_y C_z + A_y B_z C_y \rangle \\
&= B_x \langle A_y C_y + A_z C_z, 0, 0 \rangle - A_x B_x \langle 0, C_y, C_z \rangle + \\
&\quad B_y \langle 0, A_z C_z + A_x C_x, 0 \rangle - A_y B_y \langle C_x, 0, C_z \rangle + \\
&\quad B_z \langle 0, 0, A_x C_x + A_y C_y \rangle - A_z B_z \langle C_x, C_y, 0 \rangle \\
&= B_x \left\langle \vec{A} \cdot \vec{C}, 0, 0 \right\rangle - A_x B_x \langle C_x, C_y, C_z \rangle + \\
&\quad B_y \left\langle 0, \vec{A} \cdot \vec{C}, 0 \right\rangle - A_y B_y \langle C_x, C_y, C_z \rangle + \\
&\quad B_z \left\langle 0, 0, \vec{A} \cdot \vec{C} \right\rangle - A_z B_z \langle C_x, C_y, C_z \rangle \\
&= \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)
\end{aligned}$$

1.9 Unit Vectors

Consider any vector $\vec{V}$ whose magnitude is V. The ratio $\vec{V}/V$ is defined as a *unit vector*:

$$\hat{V} = \frac{\vec{V}}{V}$$

Unit vectors always have a magnitude of one, hence their name. It follows that any other vector $\vec{R}$ can be expressed as the product of a magnitude and a unit vector to indicate direction via

$$\vec{R} = R\,\hat{R}\,.$$

1.10 Basis Vectors

For a vector $\vec{V}$ of dimension N, namely

$$\vec{V} = \langle V_1, V_2, V_3, \cdots, V_N \rangle \;,$$

we may define N unit vectors according to

$$\begin{aligned} \hat{e}_1 &= \langle 1, 0, 0, \cdots, 0 \rangle \\ \hat{e}_2 &= \langle 0, 1, 0, \cdots, 0 \rangle \\ \hat{e}_3 &= \langle 0, 0, 1, \cdots, 0 \rangle \\ &\cdots \\ \hat{e}_N &= \langle 0, 0, 0, \cdots, 1 \rangle \;, \end{aligned}$$

called (normalized) *basis vectors*. That is, each dimension has its own unit vector pointing out in the positive direction. Note that all basis vectors are mutually orthogonal (perpendicular). Basis vectors never point in overlapping directions.

Using basis vectors, it follows that *any* point in the 'space' of all possible vectors can be represented by some combination of basis vectors and scale factors. Meaning, a completely equivalent expression for $\vec{V}$ is

$$\vec{V} = V_1\,\hat{e}_1 + V_2\,\hat{e}_2 + V_3\,\hat{e}_3 + \cdots + V_N\,\hat{e}_N\,.$$

Another vector $\vec{R}$ that has the same dimension as $\vec{V}$ will share the same basis vectors - there is no need to define a second set of $\hat{e}$'s.

1.11 Isolating Components

Recall that for any two basis vectors $\hat{e}_j$, $\hat{e}_k$, we have, by construction:

$$\hat{e}_j \cdot \hat{e}_k = 0 \qquad\qquad \hat{e}_k \cdot \hat{e}_k = 1$$

That is, when the two indexes match, the dot product is one because we're projecting a unit vector onto itself. When the indexes mismatch, the dot product between such orthogonal vectors is zero.

This fact can be exploited in order to 'pluck out' an individual V_k component from a vector $\vec{V}$. Starting with the expression

$$\vec{V} = V_1\,\hat{e}_1 + V_2\,\hat{e}_2 + V_3\,\hat{e}_3 + \cdots + V_N\,\hat{e}_N\;,$$

multiply across the whole equation by the $\hat{e}_k$-th basis vector as a dot product:

$$\hat{e}_k\cdot\vec{V} = V_1\,\hat{e}_k\cdot\hat{e}_1 + V_2\,\hat{e}_k\cdot\hat{e}_2 + V_3\,\hat{e}_k\cdot\hat{e}_3 + \cdots + V_N\,\hat{e}_k\cdot\hat{e}_N\;.$$

Next, notice that *each* of the dot products resolve to zero, *except* for the k-th one. It follows that:

$$\begin{aligned}
\hat{e}_k\cdot\vec{V} &= \cdots + 0 + V_k\,\hat{e}_k\cdot\hat{e}_k + 0 + \cdots \\
\hat{e}_k\cdot\vec{V} &= V_k\,\hat{e}_k\cdot\hat{e}_k \\
\hat{e}_k\cdot\vec{V} &= V_k
\end{aligned}$$

Using tighter notation, a general vector $\vec{V}$ is the sum:

$$\vec{V} = \sum_{k=1}^{N} V_k\,\hat{e}_k = \sum_{k=1}^{N}\left(\vec{V}\cdot\hat{e}_k\right)\hat{e}_k$$

Finally, note that that the symbol $\vec{V}$ doesn't automatically imply which basis is being used. It's conceivable to replace to replace all $\hat{e}_k$ with a different set $\hat{u}_k$, so long as the new basis *spans* the same space as the old basis.

2 Cartesian Plane

The *Cartesian plane* is a flat two-dimensional graphing space with two perpendicular axes that cross at an origin. Any point P on the plane is represented by an ordered pair (x, y), which lends naturally to vector notation

$$\vec{P} = \langle x, y\rangle\;,$$

where the special point $\langle 0, 0\rangle$ (zero vector) locates the origin.

2.1 Cartesian Basis Vectors

In two-dimensional Cartesian coordinates, there are two basis vectors defined as

$$\hat{x} = \langle 1, 0\rangle \qquad\qquad \hat{y} = \langle 0, 1\rangle\;.$$

It follows that any point (x, y) on the plane is represented by

$$\vec{P} = x\,\hat{x} + y\,\hat{y}\;.$$

2.2 Vector Components

In order to isolate an individual vector component, multiply the appropriate basis vector across the whole equation via dot product. For instance, to solve for x:

$$\begin{aligned} \vec{P} &= x\,\hat{x} + y\,\hat{y} \\ \hat{x}\cdot\vec{P} &= x\,(\hat{x}\cdot\hat{x}) + y\,\cancel{\hat{x}\cdot\hat{y}} \\ \hat{x}\cdot\vec{P} &= x \end{aligned}$$

In the above, we have used the fact that a basis vector projected onto itself equals one, and a basis vector projected onto any other equals zero. Using proper lingo, one might say that 'x is the $\hat{x}$-projection of $\vec{P}$'. Repeating for the y-component, we find:

$$\begin{aligned} \vec{P} &= x\,\hat{x} + y\,\hat{y} \\ \hat{y}\cdot\vec{P} &= x\,\cancel{\hat{y}\cdot\hat{x}} + y\,(\hat{y}\cdot\hat{y}) \\ \hat{y}\cdot\vec{P} &= y \end{aligned}$$

2.3 Drawing Vectors

In the Cartesian plane, a vector $\vec{V}$ looks like an arrow. We draw a vector in three steps: (i) Draw a point at the origin signifying the base of the vector. (ii) Place a second point in the plane that is V_x units from the origin in the x-direction, and V_y units form the origin in the y-direction. (iii) Connect the two points with a line, and place an arrow head at the point $\langle V_x, V_y\rangle$ pointing away from the base point.

Change of Base Point

One subtlety with drawing vectors is *you are free to translate any vector about the Cartesian plane*. That is, you can drag the arrow *anywhere*, and so long as the magnitude and direction don't change, it's still the same vector. The base point of a vector is not always the origin.

Adding Vectors Visually

You know already that two vectors can be added (or subtracted) to form a new vector, and this is done by combining corresponding pairs of components. It turns out that the 'arrow' representation of vectors avails a great shortcut for vector addition. That is: *vectors add simply by arranging them head-to-tail.* (Subtraction is a bit uglier.)

For example, consider two vectors $\vec{A} = \langle 3, 1\rangle$ and $\vec{B} = \langle 2, 4\rangle$. It's obvious that the combination $\vec{C} = \vec{A} + \vec{B} = \langle 5, 5\rangle$ by vector addition, but how do we use arrows to see this? The answer is, draw each vector with its base at the origin and the tip at $\langle A_x, A_y\rangle$, or $\langle B_x, B_y\rangle$, respectively. Next, choose a vector to move (choose $\vec{B}$), and situate its base at the tip of vector $\vec{A}$, arranging them head-to-tail. Finally, draw a new vector form the origin (base of $\vec{A}$) to the tip of $\vec{B}$. This new vector is in fact $\vec{C} = \vec{A} + \vec{B}$. Cool, right? *You can add any number of vectors this way by stacking them head-to-tail.*

Corollary

Vector additions that trace out a 'closed path' when stacked head-to-tail sum to zero.

2.4 Straight Lines

From elementary algebra, recall that any straight line in the Cartesian plane is described by

$$y = mx + b \; ,$$

where y is the vertical coordinate, x is the horizontal coordinate, m is the slope of the line, and b is the y-intercept.

We easily capture the $y = mx + b$ equation using vector apparatus. First (and easiest), define a y-intercept vector $\vec{b}$ such that

$$\vec{b} = \langle 0, b \rangle = 0 \, \hat{x} + b \, \hat{y} = b \, \hat{y} \; .$$

Next, we introduce tangent vector $\vec{T}$, parallel to the slope of the line, as

$$\vec{T} = T_x \, \hat{x} + T_y \, \hat{y} \; ,$$

where the slope is captured by the ratio

$$m = \frac{T_y}{T_x} \; .$$

By vector addition, the sum of $\vec{b}$ and $\vec{T}$ establishes the whole line, defined as the family of points $\langle x, y \rangle$:

$$\vec{P} = \langle x, y \rangle = \vec{b} + \vec{T} \; .$$

Of course, it would be incredibly cumbersome to keep readjusting the components of $\vec{T}$ while maintaining $m = T_y / T_x$. To proceed, let us divide $\vec{T}$ by its own magnitude, converting it to a unit vector $\hat{T}$. To 'pay' for this, we must introduce a dimensionless parameter α into the straight line equation, giving

$$\vec{P} = \vec{b} + \alpha \, \hat{T} \; .$$

In this form, the vectors $\vec{b}$ and $\hat{T}$ never need to change, and any point on the line is adjusted by $-\infty < \alpha < \infty$.

Although $\vec{P} = \vec{b} + \alpha \, \hat{T}$ doesn't *look* like $y = mx + b$, the familiar relation spills right out. In component form, we have

$$\begin{aligned} \langle x, y \rangle &= \langle 0, b \rangle + \langle \alpha T_x, \alpha T_y \rangle \\ &\rightarrow \\ x &= \alpha T_x \\ Y &= b + \alpha T_y \; , \end{aligned}$$

where eliminating α delivers $y = mx + b$.

Parallel Lines

A line parallel to $\vec{P} = \vec{b} + \alpha\,\hat{T}$ has a different y-intercept vector and a different dimensionless parameter β, but the same tangent:

$$\vec{P}_2 = \vec{b}_2 + \beta\,\hat{T}$$

Perpendicular Lines

Recalling that perpendicular vectors have zero dot product, it follows that two lines in the Cartesian plane whose tangent vectors are $\vec{A}$ and $\vec{B}$ are perpendicular if

$$0 = \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y \qquad\qquad \frac{B_y}{B_x} = -\frac{A_x}{A_y} \; ,$$

which you may recognize as the 'negative reciprocal' rule for perpendicular slopes:

$$m_\perp = \frac{-1}{m}$$

2.5 Circles

A circle of radius R centered at the point (h, k) in the Cartesian plane is fully described by the equation

$$(x - h)^2 + (y - k)^2 = R^2 \; .$$

We capture the same information using vectors by first defining $\vec{h}_0$ to locate the center, namely

$$\vec{h}_0 = \langle h, k \rangle \; .$$

From the point (h, k), any point on the circle is one distance R away. To arrive at a point on the circle, we may 'step away' a length R from $\vec{h}_0$ in *any* direction with mixed x- and y-components. Labeling the displacement vector $\vec{R}$, it follows that

$$\vec{R} = \langle f\,(x, y)\,, g\,(x, y) \rangle \; .$$

Computing $\vec{R} \cdot \vec{R}$, we discover a Pythagorean identity

$$R^2 = (f\,(x, y))^2 + (g\,(x, y))^2 \; .$$

Theta Parameter

Even though the functions f, g depend on both variables x, y, these variables are constrained by the (constant) radius R, along with an angular parameter θ (Greek 'theta') to specify position on the circle, where

$$0 \le \theta < 2\pi \qquad\qquad \text{or} \qquad\qquad 0^\circ \le \theta < 360^\circ \; .$$

Proceed by writing

$$\vec{R} = \langle f\,(R, \theta)\,, g\,(R, \theta) \rangle \; .$$

Or course, the radius R factors out of f and g as a scalar, giving

$$\vec{R} = R \left\langle f\left(\theta\right), g\left(\theta\right)\right\rangle \ ,$$

letting us swiftly write down a unit vector

$$\hat{r} = \frac{\vec{R}}{R} = \left\langle f\left(\theta\right), g\left(\theta\right)\right\rangle \ .$$

Cosine and Sine

The functions f, g have special names, called the *cosine*, and the *sine*, respectively. This means

$$\hat{r} = \left\langle \cos\theta, \sin\theta \right\rangle = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \ ,$$

where clearly,

$$1 = \cos^2\theta + \sin^2\theta \ .$$

Finally, the family of points (x, y) that satisfy the equation of the circle must be given by

$$\begin{aligned}
\vec{P} &= \vec{h}_0 + R\,\hat{r}\left(\theta\right) \\
\left\langle x, y\right\rangle &= \left\langle h, k\right\rangle + R\left\langle \cos\theta, \sin\theta\right\rangle \\
&\rightarrow \\
x &= h + R\cos\theta \\
y &= k + R\sin\theta \ ,
\end{aligned}$$

which reduces to the original equation

$$\left(x - h\right)^2 + \left(y - k\right)^2 = R^2 \ .$$

Unit Circle

A circle with $R = 1$ centered at the origin strictly obeys

$$x = \cos\theta \qquad\qquad y = \sin\theta \ ,$$

or

$$\vec{P} = \left\langle x, y\right\rangle = \left\langle \cos\theta, \sin\theta\right\rangle = \hat{r} \ .$$

3 Trigonometric Functions

3.1 Degrees and Radians

Degrees and radians are separate units for keeping track of the 'measure' of an angle (on a circle). A ninety-degree turn equals $\pi/2$ radians, a full $360°$ rotation is 2π radians, and so on. The word *radian*, sometimes abbreviated *rad*, is usually omitted from equations and formulas, as it is a dimensionless quantity.

Conversion between these two systems is oft necessary. Expressing degrees in terms of radians, and vice versa, looks like:

$$1° = \frac{\pi}{180}\,rad \qquad\qquad 1\,rad = \frac{180°}{\pi}$$

Angles (for our purposes, for a while at least) are part of the *real numbers*, an interval that includes any fraction or integer between $-\infty$ and $+\infty$. On the Cartesian plane, unique angles live in the interval $0 \leq \theta < 2\pi$. Any angle *outside* this region can be expressed in terms of a value *inside* this region. For example, $370°$ is essentially equivalent to $10°$.

3.2 Properties of Cosine and Sine

The *trigonometry functions* $\cos(\theta)$ and $\sin(\theta)$ each represent a decimal number for any given θ, but you'd rarely know that number off hand. Such information can be recovered from a lookup table similar to the following:

Angle (deg)	Angle (rad)	$\sin(\theta)$	$\cos(\theta)$
0	0	0	1
20	0.349	0.342	0.940
45	$0.785 = \pi/4$	0.707	0.707
70	1.22	0.940	0.342
90	$1.57 = \pi/2$	1	0

Note that θ is *always* measured from the x-axis (horizontal line) and 'curls' up in a counterclockwise direction. The range of angles corresponding to $x > 0$, $y > 0$ is within $0 \leq \theta < \pi/2$, called the *first quadrant* of the Cartesian plane. Allowing θ to surpass $\pi/2$ leads to the *second quadrant* with $x < 0$, $y > 0$, trapping θ within $\pi/2 \leq \theta < \pi$. Continuing further, we enter the *third* and *fourth* quadrants, where θ is bound by $\pi \leq \theta < 3\pi/2$, and $3\pi/2 \leq \theta < 2\pi$, respectively. Continuing the table above, we have:

Angle (deg)	Angle (rad)	$\sin(\theta)$	$\cos(\theta)$
110	1.92	0.940	-0.342
135	$2.36 = 3\pi/4$	0.707	-0.707
160	2.79	0.342	-0.940
180	$3.14 = \pi$	0	-1
200	3.40	-0.342	-0.940
225	$3.93 = 5\pi/4$	-0.707	-0.707
250	4.36	-0.940	-0.342
270	$4.71 = 3\pi/2$	-1	0
290	5.06	-0.940	0.342
315	$5.50 = 7\pi/4$	-0.707	0.707
340	5.93	-0.342	0.940
360	$6.28 = 2\pi$	0	1

The cosine and the sine are each continuous functions, which when plotted together are are essentially identical up to a *phase shift* of $\pi/2$ radians (the functions are 90° off from each other). Notice that the magnitude of each function never exceeds 1.

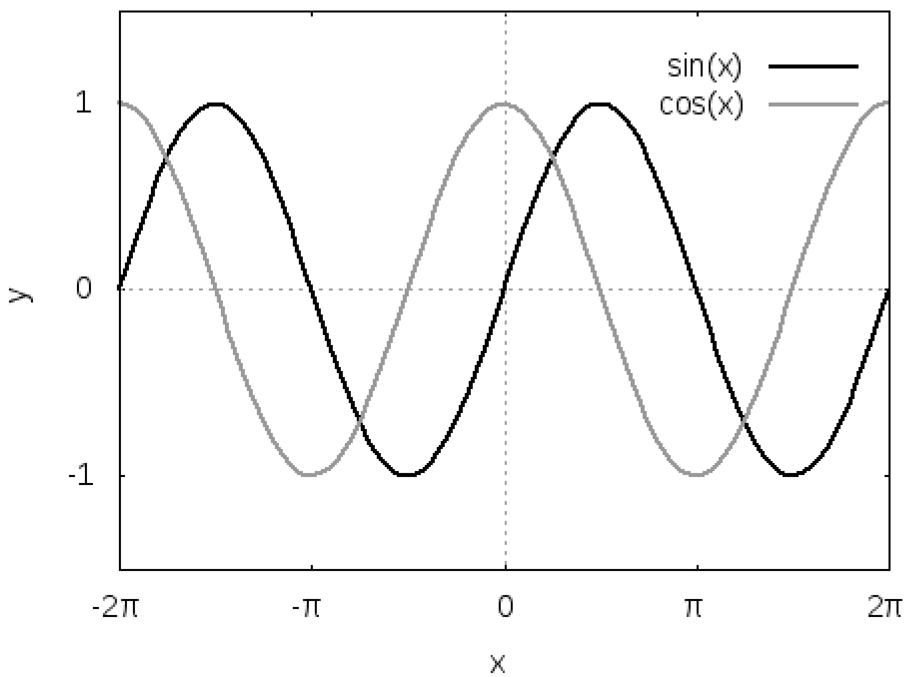

Figure 1: Sine (black) and cosine (gray) over a two-period domain.

Periodicity and Phase

Since the θ-variable is only unique on the interval $0 \leq \theta < 2\pi$, it naturally follows that trigonometric functions repeat themselves, a class known as *periodic* functions. That is:

$$\cos(\theta + 2\pi) = \cos\theta \qquad\qquad \sin(\theta + 2\pi) = \sin\theta$$

Furthermore, the cosine and sine are the same up to a phase shift $\pi/2$ as

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad\qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\,.$$

We can also jump across the circle by precisely π radians to flip the sign on either function:

$$\cos(\theta + \pi) = -\cos\theta \qquad\qquad \sin(\theta + \pi) = -\sin\theta$$

3.3 Tangent

The *tangent* function (not to be confused with a tangent line) is defined as the ratio of the sine over the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

From the definition, the tangent obeys the following periodicity and phase relations:

$$\tan(\theta + 2\pi) = \tan\theta \qquad\qquad \tan\left(\frac{\pi}{2} - \theta\right) = \tan\theta \qquad\qquad \tan(\theta + \pi) = \tan\theta$$

Because the denominator (the cosine) passes through zero, the tangent function becomes infinite at $\pm\pi/2$, $\pm 3\pi/2$, etc. as shown.

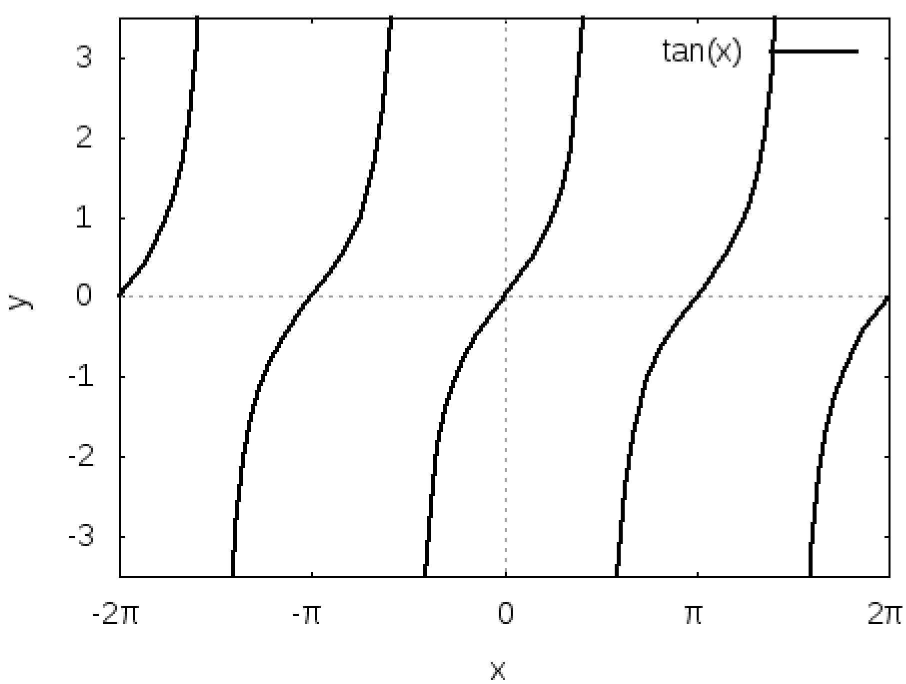

Figure 2: Tangent function.

3.4 Secant, Co-secant, Cotangent

Three more trigonometric functions are the secant, co-secant, and cotangent, defined as:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

These functions inherit the same periodicity and phase properties as the cosine, sine, and tangent, respectively.

3.5 Inverse Trigonometric Functions

For each trigonometric function F, there exists an inverse function that 'undoes' its partner. In general notation, this means

$$F^{-1}\left(F\left(\theta\right)\right) = \theta\ ,$$

where F^{-1} is the inverse of function F. As a result of the above, a similar equation emerges:

$$F\left(F^{-1}\left(\theta\right)\right) = \theta$$

The inverse of the cosine, sine, and tangent have special names as follows:

$$\arccos\left(\cos\theta\right) = \theta \qquad \cos\left(\arccos\theta\right) = \theta$$

$$\arcsin\left(\sin\theta\right) = \theta \qquad \sin\left(\arcsin\theta\right) = \theta$$

$$\arctan\left(\tan\theta\right) = \theta \qquad \tan\left(\arctan\theta\right) = \theta$$

The secant, co-secant, and cotangent also have inverse counterparts defined in analogy to those above.

4 Polar Coordinates

Consider a circle of radius r centered at the origin. We found that the family of points (x, y) that live on the circle obeys the relations

$$x = r\cos\theta \qquad y = r\sin\theta\ ,$$

where $0 \leq \theta < 2\pi$. Now, if we un-fix the radius and allow r to vary, we can get to *any* point in the Cartesian plane with the proper choice of r and θ. The r-θ notation is called *polar coordinates*. A point in the plane is located by any of:

$$\vec{P} = \langle x, y\rangle = \vec{r}\left(r,\theta\right) = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y} = r\left\langle\cos\theta,\sin\theta\right\rangle = r\,\hat{r}\left(\theta\right)$$

4.1 Polar Basis Vectors

Our study of circles in the Cartesian plane led us to write

$$\hat{r} = \langle \cos\theta, \sin\theta \rangle = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}\ ,$$

which is one basis vector for polar coordinates. Note that $\hat{r}$ is not constant, but swivels to face directly away from the origin.

There must exist a second basis vector that is (i) perpendicular to $\hat{r}$, and (ii) of unit length. Denoting this vector $\hat{\theta} = \langle A_x, A_y \rangle$, we must have

$$\hat{r} \cdot \hat{\theta} = 0 = A_x \cos\theta + A_y \sin\theta\ ,$$

which readily satisfies (i) and (ii) if we set

$$A_x = -\sin\theta \qquad\qquad A_y = \cos\theta\ ,$$

giving us

$$\hat{\theta} = \langle -\sin\theta, \cos\theta \rangle = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\ .$$

4.2 Lines and Arcs

If we choose to hold either of r or θ constant, polar coordinates generate families of lines and circles. For lines, consider any point $\vec{P} = r\,\langle \cos\theta, \sin\theta \rangle$. If we freeze θ and allow r to vary, the family of vectors $\vec{P}$ is confined to a line that bisects the whole Cartesian plane, passing through the origin.

On the other hand, if we fix the radius r and allow θ to occupy its full domain, the resultant family of points is a circle centered at the origin. A semicircular arc has a length proportional to the radius of the generating circle and the angle swept out, meaning

$$\text{Arc Length} = r\theta\ .$$

For one full revolution, set $\theta = 2\pi$ to recover the formula for the circumference of a circle.

4.3 Small Displacements and Areas

Consider a fixed point (x_0, y_0) somewhere in the Cartesian plane. A new point (x_1, y_1) placed close to the fixed point defines a small displacement vector $\Delta\vec{S}$ such that

$$\Delta\vec{S} = \langle x_1, y_1 \rangle - \langle x_0, y_0 \rangle = \langle \Delta x, \Delta y \rangle\ ,$$

or

$$\Delta\vec{S} = \Delta x\,\hat{x} + \Delta y\,\hat{y}\ .$$

The magnitude of $\Delta\vec{S}$ gives a formula for the arc length in the Cartesian plane:

$$\Delta S = \sqrt{\Delta x^2 + \Delta y^2}\ .$$

By a similar token, a small patch of area, denoted ΔA is the product of $|\Delta x|$ and $|\Delta y|$. To avoid writing absolute value symbols, we have

$$\Delta A = \sqrt{(\Delta x \Delta y)^2} \, .$$

In polar coordinates however, we don't track displacements along the $\hat{x}$- and $\hat{y}$-directions, but instead along $\hat{r}$ and $\hat{\theta}$. A small displacement in the radial direction is trivial enough to write down, namely Δr. A small displacement along $\hat{\theta}$ is ambiguous however until the radius is specified. (That is, the length introduced by increasing θ by one degree is greater when further away from the origin.) Thus, small displacements in θ correspond to a small arc length $r\Delta\theta$. It follows that the formula for small displacements in polar coordinates reads

$$\Delta\vec{S} = \Delta r \, \hat{r} + r\Delta\theta \, \hat{\theta} \, .$$

The arc length ΔS as expressed in polar coordinates therefore reads

$$\Delta S = \sqrt{\Delta r^2 + r^2 \Delta\theta^2} \, .$$

The corresponding small area patch ΔA in polar coordinates is

$$\Delta A = r\sqrt{\Delta r^2 \Delta\theta^2} \, .$$

4.4 Conversion to Polar Coordinates

The pair of equations

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

can be inverted to solve for r and θ explicitly in terms of x and y. The easy target is r, which comes from

$$r = \sqrt{x^2 + y^2} \, .$$

To solve for θ, take the ratio y/x to write

$$\frac{y}{x} = \tan\theta \, ,$$

which inverts via the arctangent function:

$$\theta = \arctan\left(\frac{y}{x}\right)$$

5 Triangles

A *triangle* is defined as a planar figure with three sides and three angles. From elementary geometry we know the interior angles of a triangle sum to 180°, also known as π radians We prove this by drawing a triangle and labeling the interior angles A, B, C as shown.

(1) Extend lines through the A- and B-corners of the triangle, and mark the vertical angles to A and B. (2) Draw a line through point C that is parallel to line AB. (3) Mark the corresponding angles to A and B on the parallel line. (4) Observe that the total arc BCA sweeps out 180 degrees on the line, thus the sum $A + B + C$ equals 180 degrees.

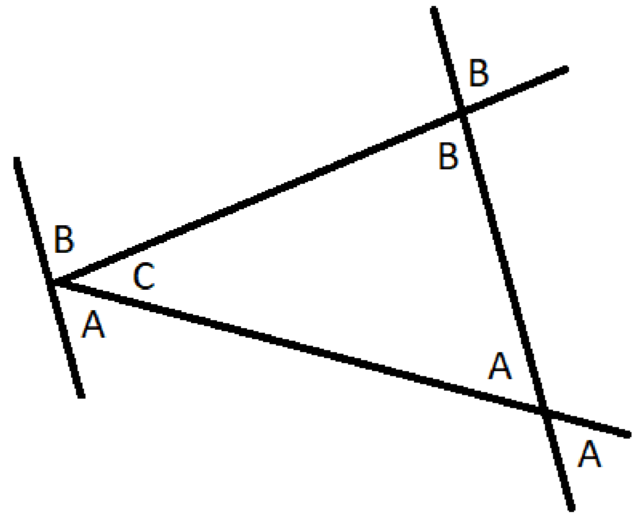

Figure 3: Triangle.

5.1 Right Triangles

A *right triangle* is any triangle that has two sides meeting at $90° = \pi/2$ radians. The intersection is labeled with a small box to denote the right angle. The side across from (not touching) the 90-degree corner is called the 'Hypotenuse'. If we label either of the non-right inner angles with the symbol θ (theta), then the side touching this is called 'Adjacent'. The side across from θ is called the 'Opposite'.

A right triangle can be held in any orientation without violating the above labels, as shown in Figure 4. Note that the angle interior angle θ is strictly limited to the domain $(0, \pi/2)$.

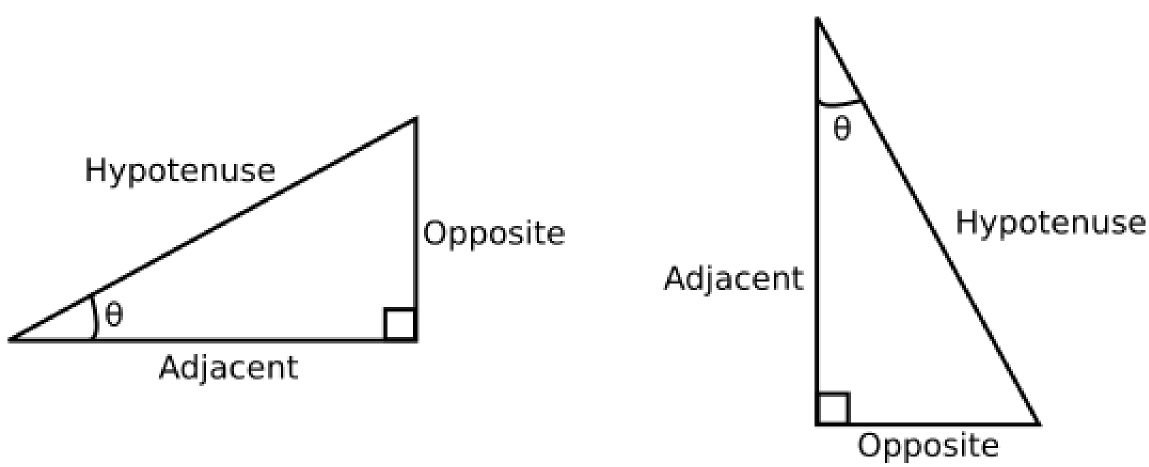

Figure 4: Two right triangles with labels indicating the Hypotenuse, Opposite, and Adjacent, based on interior angle θ.

Pythagorean Theorem

Right triangles lend themselves to a very useful algebraic identity known as the *Pythagorean theorem*. Using our new terminology for the sides of the triangle, the theorem reads

$$(Opposite)^2 + (Adjacent)^2 = (Hypotenuse)^2 \ .$$

This should already be familiar and has been used liberally thus far. In combined terminology we may say that the hypotenuse of a right triangle is the magnitude of the sum of the opposite and adjacent vectors.

SohCahToa

Any position vector $\vec{P}$ on a circle of radius R is the hypotenuse of a right triangle with the adjacent on the x-axis and the opposite parallel to the y-axis as shown in Figure 5.

By convention, we always place θ such that the interior angle 'opens up' from the x-axis, increasing in the counter-clockwise direction. Negative angles are also allowed: simply 'unwind' θ in the clockwise direction.

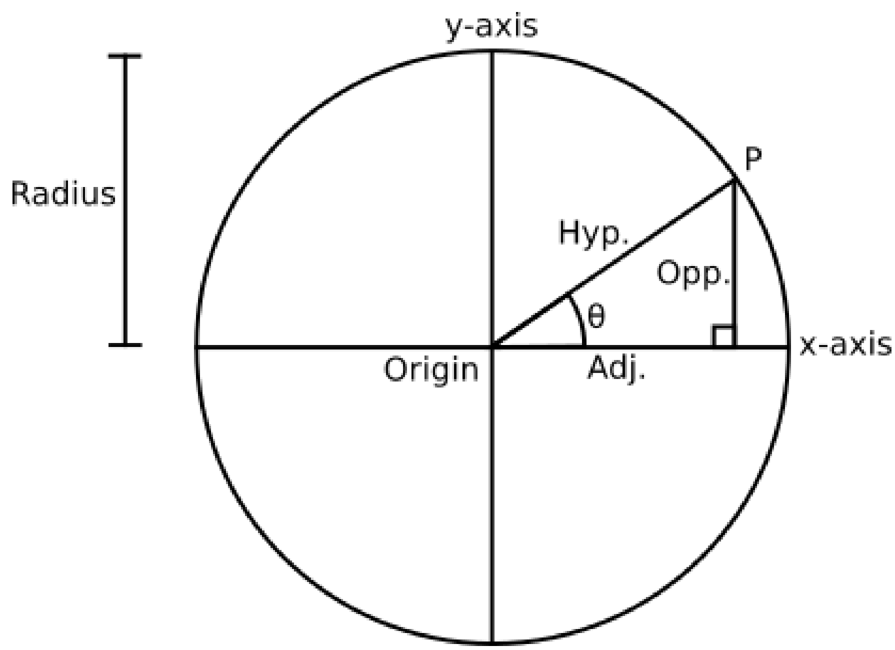

Figure 5: A right triangle embedded in a circle.

Recall that the polar representation of a circle is captured in

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \frac{y}{x} = \tan\theta \ .$$

By embedding a right triangle in a circle, the x-coordinate becomes the 'adjacent' of the triangle, and meanwhile the y-coordinate becomes the 'opposite' side. Solving for $\sin\theta$, $\cos\theta$, $\tan\theta$ respectively, we have:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Taking the first letter of each label, you can pick out a 'name'

$$SohCahToa$$

that should help identify the sides of a right triangle while problem solving.

5.2 Non-Right Triangles

Consider two vectors $\vec{A}$ and $\vec{B}$, each having arbitrary magnitude and direction, both drawn with their base point at the origin. A third vector $\vec{C}$ extends from the tip of $\vec{A}$ to the tip of $\vec{B}$, creating a triangle, represented by

$$\vec{B} - \vec{A} = \vec{C}\,.$$

One perfectly legal move is to 'square' each side of the equation by dotting it with itself as

$$\begin{aligned}\left(\vec{B} - \vec{A}\right) \cdot \left(\vec{B} - \vec{A}\right) &= \vec{C} \cdot \vec{C} \\ \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2\vec{A} \cdot \vec{B} &= \vec{C} \cdot \vec{C} \\ A^2 + B^2 - 2\vec{A} \cdot \vec{B} &= C^2\,.\end{aligned}$$

The special case for right triangles has $\vec{A} \cdot \vec{B} = 0$, in which case the above reduces to the Pythagorean theorem.

Law of Cosines

The relation

$$A^2 + B^2 - 2\vec{A} \cdot \vec{B} = C^2$$

is called the *law of cosines*, which is a generalization of the Pythagorean theorem for non-right triangles.

6 Vectors and Geometry

6.1 Angle-Sum Formulas

One can always express an angle θ as the sum of two smaller angles α and β, as sketched out in Figure 6 on the unit circle. Note first that the line indicated by β projects down onto the line indicated by α, allowing $\cos(\beta)$ and $\sin(\beta)$ to be identified as components of a right triangle. Projecting these onto the x- and y- axes, we discover the two angle-sum formulas:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Moving on to $\tan(\alpha \pm \beta)$, there is no clever argument from geometry to discover the result, so we need brute force (using the two above equations):

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

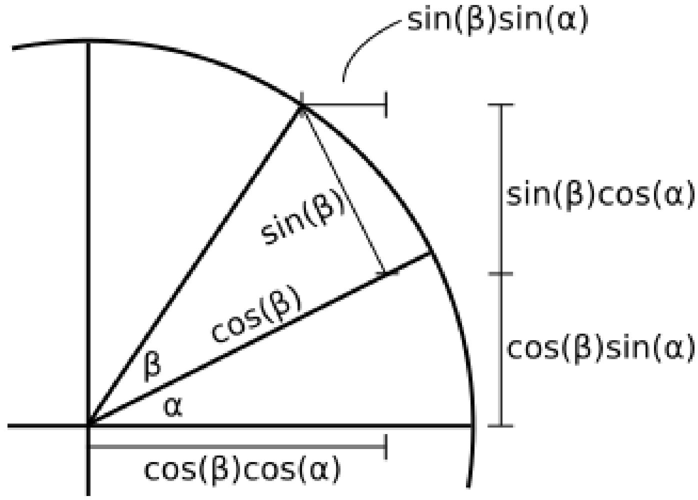

Figure 6: Depiction of $\theta = \alpha + \beta$ on the unit circle.

6.2 Dot and Cross Product Revisited

Consider two vectors $\vec{A}$ and $\vec{B}$ of non-equal magnitude and different orientations:

$$\vec{A} = A\left\langle \cos\theta_A, \sin\theta_B \right\rangle \qquad\qquad \vec{B} = B\left\langle \cos\theta_B, \sin\theta_B \right\rangle$$

By calculating the dot product $\vec{A}\cdot\vec{B}$, the result is

$$\vec{A}\cdot\vec{B} = AB\left(\cos\theta_A\cos\theta_B + \sin\theta_A\sin\theta_B\right) \ ,$$

which by the angle-sum formula above, reduces to

$$\vec{A}\cdot\vec{B} = AB\cos\left(\theta_A - \theta_B\right) \ .$$

Corollary

The dot product between two vectors is the product of their magnitudes and the cosine of the angle between them:

$$\vec{A}\cdot\vec{B} = AB\cos\phi \qquad\qquad \phi = \theta_A - \theta_B$$

To explicitly solve for ϕ, make use of the arccos function:

$$\phi = \arccos\left(\frac{\vec{A}\cdot\vec{B}}{AB}\right) = \arccos\left(\frac{A_xB_x + A_yB_y}{AB}\right)$$

Meanwhile, the cross product between the same two vectors $\vec{A}$ and $\vec{B}$ (with $\theta_A > \theta_B$) is written

$$\left|\vec{A}\times\vec{B}\right| = AB\left(\cos\theta_A\sin\theta_B - \sin\theta_A\cos\theta_B\right) \ ,$$

which is equivalent to

$$\left|\vec{A}\times\vec{B}\right| = AB\sin\left(\theta_A - \theta_B\right) \ .$$

Corollary

The cross product between two vectors is the product of their magnitudes and the sine of the angle between them:

$$\left|\vec{A} \times \vec{B}\right| = AB \sin\phi \qquad\qquad \phi = \theta_A - \theta_B$$

To explicitly solve for ϕ, make use of the arcsin function:

$$\phi = \arcsin\left(\frac{\left|\vec{A} \times \vec{B}\right|}{AB}\right) = \arcsin\left(\frac{A_x B_y - A_y B_x}{AB}\right)$$

Area of a Parallelogram

One useful interpretation of the quantity $AB \sin\phi$ is the area of a parallelogram of base B and height $h = A \sin\phi$. For the right-angle case $\phi = \pi/2$, the parallelogram becomes a rectangle of area AB.

Right Hand Rule

There is a trick that allows one to know the direction of $\vec{A} \times \vec{B}$ without resorting to the full three-dimensional formula, and this is the (oft-dreaded) *right hand rule*. To know the direction of the vector $\vec{A} \times \vec{B}$, the steps are as follows:

1. On your right hand: point your thumb, index finger, and middle finger out in perpendicular directions.
2. Let your middle finger be vector $\vec{A}$, assign vector $\vec{B}$ to your index finger.
3. If you did the two previous steps correctly, then your thumb points along vector $\vec{C}$.

Note that if you use your left hand, or accidentally swap $\vec{A}$ and $\vec{B}$, then the resultant vector will be pointing the opposite way. This fact is contained in the identity

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \,.$$

6.3 Rotations

Rotated Vectors

Consider a vector $\vec{V} = \langle V_x, V_y \rangle$ of magnitude V oriented at angle θ. If we leave the magnitude unchanged but change the orientation by some amount ϕ, a new vector $\vec{W}$ emerges:

$$\vec{W} = V \left\langle \cos\left(\theta + \phi\right), \sin\left(\theta + \phi\right) \right\rangle$$

Using the angle-sum formulas, the trigonometric terms fragment apart:

$$\vec{W} = V \left\langle \cos\theta \cos\phi - \sin\theta \sin\phi, \sin\theta \cos\phi + \cos\theta \sin\phi \right\rangle$$
$$\vec{W} = \left\langle V_x \cos\phi - V_y \sin\phi, V_y \cos\phi + V_x \sin\phi \right\rangle$$

In component form, the elements W_x, W_y are

$$W_x = V_x \cos\phi - V_y \sin\phi \qquad\qquad W_y = V_y \cos\phi + V_x \sin\phi \, .$$

Of course, a tighter notation uses a rotation matrix to act on $\vec{V}$, resulting in $\vec{W}$:

$$\begin{bmatrix} W_x \\ W_y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix}$$

Corollary

A zero-degree rotation, i.e., one that leaves the original vector unchanged, invokes a diagonal matrix of ones called the *identity matrix*:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rotated Coordinate Systems

Suppose we begin with the basis vectors $\hat{x}$, $\hat{y}$ and rotate each by an angle ϕ into a new perpendicular pair of basis vectors $\hat{u}$, $\hat{v}$. Starting with

$$\hat{x} = \langle 1, 0 \rangle \qquad\qquad \hat{y} = \langle 0, 1 \rangle \ ,$$

we employ vector rotation apparatus to derive

$$\hat{u} = \langle \cos\phi, \sin\phi \rangle \qquad\qquad \hat{v} = \langle -\sin\phi, \cos\phi \rangle \ ,$$

or

$$\hat{u} = \cos\phi \, \hat{x} + \sin\phi \, \hat{y} \qquad\qquad \hat{v} = -\sin\phi \, \hat{x} + \cos\phi \, \hat{y} \, .$$

The task on hand is to express the components of any vector $\vec{V}$ in terms of the new basis vectors $\hat{u}$, $\hat{v}$. Proceed by solving for $\hat{x}$, $\hat{y}$ in terms of $\hat{u}$, $\hat{v}$, which can be achieved in a number of ways. Here we choose

$$\hat{x} = (\hat{x} \cdot \hat{u}) \ \hat{u} + (\hat{x} \cdot \hat{v}) \ \hat{v} \qquad\qquad \hat{y} = (\hat{y} \cdot \hat{u}) \ \hat{u} + (\hat{y} \cdot \hat{v}) \ \hat{v}$$

$$\hat{x} = \cos\phi \, \hat{u} - \sin\phi \, \hat{v} \qquad\qquad \hat{y} = \sin\phi \, \hat{u} + \cos\phi \, \hat{v} \ ,$$

which does the job. Explicitly, this means a vector $\vec{V} = \langle V_x, V_y \rangle$ can be written in the new basis as

$$\vec{V} = V_x \, \hat{x} + V_y \, \hat{y} = V_x \left(\cos\phi \, \hat{u} - \sin\phi \, \hat{v} \right) + V_y \left(\sin\phi \, \hat{u} + \cos\phi \, \hat{v} \right) \ ,$$

or in simplified form,

$$\vec{V} = \left(V_x \cos\phi + V_y \sin\phi \right) \ \hat{u} + \left(-V_x \sin\phi + V_y \cos\phi \right) \ \hat{v} \, .$$

The parenthesized terms have the appearance of a rotated vector $\vec{V}'$ 'backwards' through the angle $-\phi$. Condensing the notation further, we have

$$\vec{V} = V_u \, \hat{u} + V_v \, \hat{v} \ ,$$

where the V_u, V_v components are generated from a modified rotation matrix

$$\begin{bmatrix} V_u \\ V_v \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} \ .$$

7 Trigonometric Identities

7.1 Founding Identities

The familiar players of trigonometry are the cosine, sine, tangent

$$\cos\theta \qquad \sin\theta \qquad \tan\theta = \frac{\sin\theta}{\cos\theta},$$

along with their counterparts, the secant, co-secant, and cotangent:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

It's not news that the fundamental identity

$$\cos^2\theta + \sin^2\theta = 1$$

always holds, which yields two more identities if we divide by $\cos^2\theta$ and $\sin^2\theta$, respectively:

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

By considering triangles embedded in the unit circle, we previously deduced the angle-sum formulas:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\tan(\alpha\pm\beta) = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha\pm\beta)} = \frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$$

Starting with the results (re)iterated here, a myriad of *trigonometric identities* can be derived.

7.2 Product Formulas

Take $\sin(\alpha+\beta)$ and add/subtract to/from $\sin(\alpha-\beta)$ to derive two *product formulas.* Repeat by replacing sin with cos to produce two more.

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$
$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$
$$2\sin(\alpha)\sin(\beta) = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$

7.3 Double-Angle Formulas

Let $\alpha = \beta = \theta$ in the product formulas to derive the *double-angle formulas*:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

7.4 Half-Angle Formulas

The $\cos(2\theta)$ equation can be mixed around to yield the *half-angle formulas*. First replace the $\sin^2(\theta)$ by $1 - \cos^2(\theta)$, and then re-factor the θ variable by letting $\theta \to \theta/2$, giving

$$\cos(\theta) = 2\cos^2(\theta/2) - 1 \, .$$

Going from there, the *half-angle formulas* pop out:

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2} + \frac{\cos(\theta)}{2}}$$

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2} - \frac{\cos(\theta)}{2}}$$

7.5 Superposition Relationships

The *superposition relationships* are useful in applied physics, particularly when summing waveforms. Begin by writing out the product $\sin(\alpha + \beta)\cos(\alpha - \beta)$, and simplify like mad:

$$\begin{aligned}
\sin(\alpha + \beta)\cos(\alpha - \beta) &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&= (\sin\alpha\cos\alpha)\left(\sin^2\beta + \cos^2\beta\right) + (\sin\beta\cos\beta)\left(\sin^2\alpha + \cos^2\alpha\right) \\
&= \frac{\sin(2\alpha)}{2} + \frac{\sin(2\beta)}{2}
\end{aligned}$$

Note that several trig identities have been used without being mentioned (you should get good at seeing this). Re-factor the α and β variables and arrive at out first result

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \, .$$

One down, three to go. Luckily we can let $\beta \to -\beta$ to get the next equation for free:

$$\sin(\alpha) - \sin(\beta) = 2\sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right) \, .$$

Proceeding in a similar spirit, start with $\cos(\alpha+\beta)\cos(\alpha-\beta)$ and do it all again:

$$\begin{aligned}\cos(\alpha+\beta)\cos(\alpha-\beta) &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= \cos^2\alpha\cos^2\beta + (1-1)(\sin\alpha\cos\alpha\sin\beta\cos\beta) - \sin^2\alpha\sin^2\beta \\ &= \frac{\cos^2\beta - \sin^2\alpha}{2} + \frac{\cos^2\alpha - \sin^2\beta}{2} \\ &= \frac{\cos(2\alpha)}{2} + \frac{\cos(2\beta)}{2}\end{aligned}$$

Don't be *too* cavalier about following the algebra here. If the third step didn't seem to follow smoothly from the second one, think harder! As before, re-factor the α and β variables to arrive at our third result

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) .$$

Lastly, we have to write out $\sin(\alpha+\beta)\sin(\alpha-\beta)$, which goes similarly to the previous calculation:

$$\begin{aligned}\sin(\alpha+\beta)\sin(\alpha-\beta) &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta) \\ &= \sin^2\alpha\cos^2\beta + (1-1)(\sin\alpha\cos\alpha\sin\beta\cos\beta) - \cos^2\alpha\sin^2\beta \\ &= \frac{-\cos^2\alpha + \sin^2\alpha}{2} + \frac{\cos^2\beta - \sin^2\beta}{2} \\ &= -\frac{\cos(2\alpha)}{2} + \frac{\cos(2\beta)}{2}\end{aligned}$$

Refactoring α and β, we finally have

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) .$$

7.6 Inverse Identities

Begin with the trivial statement

$$\cos(\arccos x) = x ,$$

and square each side to get

$$\cos^2(\arccos x) = 1 - \sin^2(\arccos x) = x^2 ,$$

and separate terms:

$$\sin(\arccos x) = \sqrt{1-x^2}$$

Divide through by x to get a similar identity:

$$\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x}$$

Continuing in this spirit, there remain 12 non-trivial equations in addition to the two just found. However, we can use a trick to keep such identities at the ready. Notice first the two preceding identities fall out easily if we draw a right triangle with a hypotenuse of 1, with the adjacent obeying $\cos(\theta) = x$. Naturally the argument θ equals $\arccos(x)$, and the Opposite obeys $\sin(\theta) = \sqrt{1-x^2}$. Further identities are generated by a similar procedure, where the general idea is to start with a right triangle and assign *any* two sides the lengths 1 and x.

8 Calculus and Trigonometry

8.1 Tangent Vector on a Curve

Consider a pair of parametrized functions

$$x = x(\beta) \qquad\qquad y = y(\beta) \ ,$$

which together make a position vector

$$\vec{r}(\beta) = x(\beta)\ \hat{x} + y(\beta)\ \hat{y} \ .$$

At any given point $\vec{r}_0$ on the curve, the curve itself is locally approximated by a straight line

$$\vec{r} = \vec{r}_0 + \alpha\,\hat{t} \ ,$$

where $\hat{t}$ is a unit tangent vector at the point $\vec{r}_0$ and α is a small parameter.

Recall from our study of straight lines that the ratio t_y/t_x is the slope of the curve at $\vec{r}_0$. The 'rise over run' is equivalent to the ratio $\sin\theta/\cos\theta = \tan\theta$, meaning

$$\frac{t_y}{t_x} = \frac{\sin\theta}{\cos\theta} = \tan\theta = y'(x) \qquad\qquad t_y = \tan\theta\, t_x \ ,$$

where the derivative $y'(x)$ generalizes the slope m. As a normalized unit vector, $\hat{t}$ must be

$$\hat{t} = \frac{\hat{x} + \tan\theta\,\hat{y}}{\sqrt{1+\tan^2\theta}} = \frac{\hat{x} + y'\,\hat{y}}{\sqrt{1+(y')^2}}$$

8.2 Small Angles

Consider the trig functions $\sin\theta$ and $\cos\theta$ on the unit circle (where else?). For very small angles $\theta \to \Delta\theta$, show that the cosine is approximately 1, and that the sine is approximately equal to $\Delta\theta$. The easy case is the cosine, as when $\theta \to 0$, the function is undoubtedly very close to one.

On the other hand, it would be too blunt to assume $\sin(\theta \to 0) \approx 0$. On the unit circle, very small θ implies a very skinny right triangle of hypotenuse 1, adjacent ≈ 1, and opposite $\approx \theta$. In conclusion, we find

$$\cos(\theta \to 0) = 1 \qquad\qquad \sin(\theta \to 0) = \theta \ .$$

Next, take the the angle-sum formulas

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \ ,$$

and let us interpret $\alpha = \theta$ as a base point on the unit circle, where $\beta = \Delta\theta$ is a small angular displacement away from θ. Using what we know about small angles, find:

$$\sin(\theta+\Delta\theta) \approx \sin(\theta) + \Delta\theta\cos(\theta)$$

$$\cos(\theta+\Delta\theta) \approx \cos(\theta) - \Delta\theta\sin(\theta)$$

Example

Knowing that $\sin 20° = 0.342$ and $\cos 20° = 0.940$, use the above to estimate $\sin 22°$ and $\cos 22°$.

$$\sin(22°) \approx (0.342) + \frac{(22-20)\,\pi}{180}(0.940) \approx 0.375$$
$$\cos(22°) \approx (0.940) - \frac{(22-20)\,\pi}{180}(0.342) \approx 0.928$$

8.3 Derivative Formulas

Using the definition of the derivative

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

along with the small-$\Delta\theta$ equations for the sine and cosine, we discover derivative formulas for each:

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$

Applying the same derivative formula (and anything it generates), derivatives of the remaining trigonometric functions emerge:

$$\frac{d}{dx}\tan x = 1+\tan^2 x = \sec^2 x$$
$$\frac{d}{dx}\cot x = -1-\cot^2 x = -\csc^2 x$$
$$\frac{d}{dx}\sec x = \frac{\sin x}{\cos^2 x} = \sec x\tan x$$
$$\frac{d}{dx}\csc x = \frac{-\cos x}{\sin^2 x} = -\csc x\cot x$$

8.4 Anti-Derivative Formulas

Anti-differentiate the derivative equations above to write two integrals:

$$\int \cos x \, dx = \sin x + C$$
$$\int \sin x \, dx = -\cos x + C$$

Using basic rules of calculus along with the integrals above we may derive the following (omitting the integration constant for now on):

$$\int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x} = -\int \frac{d(\cos x)}{\cos x} = -\ln|\cos x|$$
$$\int \cot x \, dx = \int \frac{\cos x \, dx}{\sin x} = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x|$$
$$\int \sec x \, dx = \int \frac{\cos x \, dx}{1-\sin^2 x} = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| = \ln|\sec x + \tan x|$$
$$\int \csc x \, dx = \int \frac{\sin x \, dx}{1-\cos^2 x} = \frac{-1}{2}\ln\left|\frac{\cos x + 1}{\cos x - 1}\right| = -\ln|\csc x + \cot x|$$

8.5 Reduction Formulas

We next use the integration by parts equation

$$\int u \, dv = uv\Big| - \int v \, du$$

to derive the *reduction formulas*.

Let:

$$u = \sin^{m-1} x \qquad dv = \sin x \, dx$$
$$\int \sin^m x \, dx = \frac{-1}{m}\cos x \sin^{m-1} x + \frac{m-1}{m}\int \sin^{m-2} x \, dx$$

Let:

$$u = \cos^{m-1} x \qquad dv = \cos x \, dx$$
$$\int \cos^m x \, dx = \frac{1}{m}\sin x \cos^{m-1} x + \frac{m-1}{m}\int \cos^{m-2} x \, dx$$

Let:

$$u = \sec^{m-2} x \qquad dv = \sec^2 x \, dx$$
$$\int \sec^m x \, dx = \frac{1}{m-1}\tan x \sec^{m-2} x + \frac{m-2}{m-1}\int \sec^{m-2} x \, dx$$

Let:

$$u = \sin^{m-1} x \qquad dv = \cos^n x \sin x \, dx$$
$$\int \sin^m x \cos^n x \, dx = \frac{-1}{m+n}\cos^{n+1} x \sin^{m-1} x + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x \, dx$$

8.6 Mixed Integrals

For integer m and n such that $m \neq n$, let

$$u = \sin mx \qquad\qquad dv = \sin nx \, dx$$

or use

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha + \beta) - \cos(\alpha - \beta)$$

to derive:

$$\begin{aligned} \int \sin mx \sin nx \, dx &= \left(\frac{1}{1 - m^2/n^2} \right) \left(\frac{m}{n^2} \cos mx \sin nx - \frac{1}{n} \sin mx \cos nx \right) \\ &= \frac{\sin((m-n)x)}{2(m-n)} - \frac{\sin((m+n)x)}{2(m+n)} \end{aligned}$$

For integer m and n such that $m \neq n$, let

$$u = \cos mx \qquad\qquad dv = \cos nx \, dx$$

or use

$$2 \cos(\alpha) \cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$

to derive:

$$\begin{aligned} \int \cos mx \cos nx \, dx &= \left(\frac{1}{1 - m^2/n^2} \right) \left(\frac{-m}{n^2} \sin mx \cos nx + \frac{1}{n} \cos mx \sin nx \right) \\ &= \frac{\sin((m-n)x)}{2(m-n)} + \frac{\sin((m+n)x)}{2(m+n)} \end{aligned}$$

For integer m and n such that $m \neq n$, let

$$u = \sin mx \qquad\qquad dv = \cos nx \, dx$$

to derive:

$$\begin{aligned} \int \sin mx \cos nx \, dx &= \left(\frac{1}{1 - m^2/n^2} \right) \left(\frac{m}{n^2} \cos mx \cos nx + \frac{1}{n} \sin mx \sin nx \right) \\ &= -\frac{\cos((m-n)x)}{2(m-n)} - \frac{\cos((m+n)x)}{2(m+n)} \end{aligned}$$

8.7 Order-Two Cases

Let

$$u = \sin ax \qquad\qquad dv = \sin ax \, dx$$

to derive:

$$\int \sin^2 ax \, dx = -\frac{\sin ax \cos ax}{2a} + \frac{ax}{2}$$

Let

$$u = \cos ax \qquad\qquad dv = \cos ax \, dx$$

to derive:

$$\int \cos^2 ax\, dx = \frac{\sin ax \cos ax}{2a} + \frac{ax}{2}$$

Derive

$$\int \sin ax \cos ax\, dx = \frac{\sin^2 ax}{2a}$$

by noticing

$$\int \sin ax \cos ax\, dx = \frac{1}{a}\int \sin ax\, d(\sin ax) = \frac{1}{a}\int q\, dq = \frac{q^2}{2a} = \frac{\sin^2 ax}{2a}\,.$$

8.8 Orthogonality

Trigonometric functions obey a variety of *orthogonality relations*, which follow from the results derived above:

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0$$
$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0$$
$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0$$
$$\int_{-\pi}^{\pi} \sin ax \cos ax\, dx = 0$$
$$\int_{-\pi}^{\pi} \sin^2 ax\, dx = \pi$$
$$\int_{-\pi}^{\pi} \cos^2 ax\, dx = \pi$$

9 Problems

9.1 Calculating Pi from Nested Radicals

Introduction

A quarter-circle can be systematically covered by non-overlapping triangles of decreasing area until, in the infinite limit, the whole shape is covered. Working on a unit circle, we set lengths $\overline{OA}$ and $\overline{OB}$ equal to one.

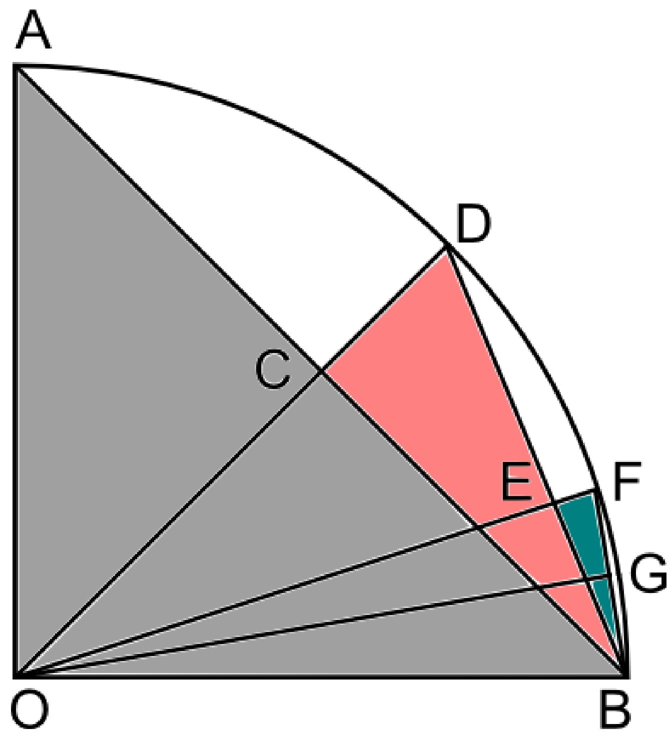

Figure 7: Quarter-unit circle partially covered by triangles.

Zero-Order Triangle (1)

The largest triangle that fits in the quarter-unit circle is AOB, whose area is clearly $1/2$. To start a pattern though, we'll write this as:

$$A_0 = \frac{1}{2}(1)(1) = \frac{1}{2}$$

Since the lines $\overline{OA}$, $\overline{OB}$ are perpendicular, let us define two unit vectors

$$\hat{i} = \overline{OB} \qquad \hat{j} = \overline{OA}\,,$$

and the line $\overline{AB}$ is the hypotenuse of AOB, which implies a non-unit vector

$$\vec{h}_0 = \overline{AB} = \hat{i} - \hat{j}\,.$$

First-Order Triangles (2)

The triangles DCA and DCB are identical by symmetry, so we may focus on DCB and remember to multiply its area by two to cover the quarter-circle. To denote the sides of DCB, first define two vectors

$$\hat{x}_1 = \overline{OD} \qquad\qquad \vec{x}_1 = \overline{OC}\ ,$$

which differ only in length. Remember that any vector extending from O and touching the arc of the circle has length one, and is denoted as a unit vector. Any vector longer or shorter gets no hat, and wears an arrow symbol. The hypotenuse may be represented as

$$\vec{h}_1 = \overline{DB}\ .$$

To proceed, we need to write the first-order vectors in the already-established notation, which means solving for $\hat{x}_1$, $\vec{x}_1$, $\vec{h}_1$ in terms of $\hat{i}$ and $\hat{j}$. By inspection of the diagram, observe that $\overline{OC}$ bisects triangle AOB, so we find:

$$\vec{x}_1 = \frac{\hat{i}+\hat{j}}{2} \qquad\qquad \hat{x}_1 = \frac{\hat{i}+\hat{j}}{\sqrt{2}} \qquad\qquad \vec{h}_1 = \hat{i} - \hat{x}_1$$

Finally, observe that the area of DCB is

$$A_{DCB} = \frac{\overline{CB}\cdot\overline{CD}}{2} = \frac{1}{2}\frac{|\vec{h}_0|}{2}\left(1 - |\vec{x}_1|\right) = \frac{1}{2}\frac{\sqrt{\vec{h}_0\cdot\vec{h}_0}}{2}\left(1 - \sqrt{\vec{x}_1\cdot\vec{x}_1}\right)\ ,$$

which is easily reduced to a number. Recalling though that the quarter-circle contains two copies of the area DCB, let is write the first-order area A_1 as

$$A_1 = 2\cdot A_{DCB} = 2\cdot\frac{1}{2}\frac{\sqrt{2}}{2}\left(1 - \sqrt{\frac{1}{2}}\right) = -\frac{1}{2} + \frac{1}{\sqrt{2}}$$

Second-Order Triangles (4)

In the diagram above, the triangle FEB is one of four identical copies, so the goal now is to get a number for the area A_{FEB}, and then construct the second-order area $A_2 = 4\cdot A_{FEB}$. We need two more vectors

$$\hat{x}_2 = \overline{OF} \qquad\qquad \vec{x}_2 = \overline{OE}\ ,$$

along with a hypotenuse

$$\vec{h}_2 = \overline{FB}\ .$$

A reliable pattern begins to emerge here. Note that in order to 'get to' point E from the origin, the vector sum of $\overline{OD}$ and $\overline{DE}$ stays along lines already defined, so we easily write

$$\vec{x}_2 = \hat{x}_1 + \frac{1}{2}\vec{h}_1$$

which means we 'go to the top of the previous triangle, and walk halfway down the hypotenuse'. Coming up with the unit vector $\hat{x}_2$ is a small chore, which resolves to

$$\hat{x}_2 = \left(\frac{2\sqrt{2}}{3} - \frac{1}{3} \right) \left(\hat{i} + \hat{j} \right) \ ,$$

and of course, the hypotenuse vector in terms of $\hat{i}$, $\hat{j}$, is

$$\vec{h}_2 = \hat{i} - \hat{x}_2 \ .$$

The second-order area calculation looks just like the previous area calculation with the vector labels shifted up by one:

$$A_2 = 4 \cdot A_{FEB} = 4 \cdot \frac{1}{2} \frac{\sqrt{\vec{h}_1 \cdot \vec{h}_1}}{2} \left(1 - \sqrt{\vec{x}_2 \cdot \vec{x}_2} \right) = -\frac{1}{\sqrt{2}} + \sqrt{2 - \sqrt{2}}$$

Third-Order Triangles (8)

By now, we should be able to proceed by pattern alone. There are eight third-order triangles in the quarter circle, and meanwhile three vectors

$$\vec{x}_3 = \hat{x}_2 + \frac{1}{2}\vec{h}_2 \qquad \hat{x}_3 = \frac{\vec{x}_3}{\sqrt{\vec{x}_3 \cdot \vec{x}_3}} \qquad \vec{h}_3 = \hat{i} - \hat{x}_3$$

determine their size and orientation. Any variable with a 3-subscript traces back to one with a 2-subscript, all the way back to $\hat{i}$, $\hat{j}$. The third-order area, after a bit of algebra, resolves to

$$A_3 = 8 \cdot \frac{1}{2} \frac{\sqrt{\vec{h}_2 \cdot \vec{h}_2}}{2} \left(1 - \sqrt{\vec{x}_3 \cdot \vec{x}_3} \right) = -\sqrt{2 - \sqrt{2}} + 2\sqrt{2 - \sqrt{2 + \sqrt{2}}} \, .$$

Any-Order Triangles

Evidently, for a triangle of order n, there are 2^n copies of it that cover the quarter-unit circle. The sides are determined by

$$\vec{x}_{n+1} = \hat{x}_n + \frac{1}{2}\vec{h}_n \qquad \vec{h}_n = \hat{i} - \hat{x}_n \ ,$$

and the area is

$$A_n = 2^n \cdot \frac{1}{2} \frac{\sqrt{\vec{h}_{n-1} \cdot \vec{h}_{n-1}}}{2} \left(1 - \sqrt{\vec{x}_n \cdot \vec{x}_n} \right) \ .$$

Applying this to the next case of $n = 4$, one finds

$$A_4 = -2\sqrt{2 - \sqrt{2 + \sqrt{2}}} + 4\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \, ,$$

which is one heck of a job to do by hand.

Finding a Pattern

Each set of triangles has a total area that more-and-more deeply embeds the square root of 2. Listing these in a row, we have

$$A_0 = \frac{1}{2}$$
$$A_1 = -\frac{1}{2} + \frac{1}{\sqrt{2}}$$
$$A_2 = -\frac{1}{\sqrt{2}} + \sqrt{2 - \sqrt{2}}$$
$$A_3 = -\sqrt{2 - \sqrt{2}} + 2\sqrt{2 - \sqrt{2 + \sqrt{2}}}$$
$$A_4 = -2\sqrt{2 - \sqrt{2 + \sqrt{2}}} + 4\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

Interestingly, each A_n (beyond A_0) contains a positive term and a negative term, where the negative term just happens to negate the positive term from A_{n-1}. While this was observed in a 'brute force' sense for trivial cases, we may argue by induction that the grand sum of each A_n cancels all terms except the final positive term. Evidently then, we find

$$A = \sum_{n=0}^{N} A_n = \frac{2^n}{4}\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}} \qquad (N \text{ square roots}) \ ,$$

which is the area of an N-sided polygon approximating the circle.

Taking the Limit

If we finally let N approach infinity, the term 2^N approaches infinity very quickly, where meanwhile the square root term approaches $\sqrt{2-2}$. This seems like a dead end, as infinity is being multiplied by zero. However, we just carefully covered the quarter-unit circles, so is should follow that $4A$ is the area of the whole unit circle, also known as π... And indeed this is true, the infinite expression

$$\pi = \lim_{N\to\infty} 2^N \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}} \qquad (N \text{ square roots})$$

converges to $3.1415926535... = \pi$.

BASIC Code Example

```
DIM b AS DOUBLE
n = 20
a = 2 ^ n
```

```
b = SQR(2)
FOR k = 1 TO n - 2
    b = SQR(2 + b)
NEXT
b = SQR(2 - b)
PRINT a * b

3.14159...
```

9.2 Capstan Equation

Problem 1

Consider a rope wrapped around a circular pole having radius R and coefficient of friction μ with the rope. Use Newton's third law in the 'radial' and 'tangential' directions to prove that the frictional force grows exponentially with the number of turns.

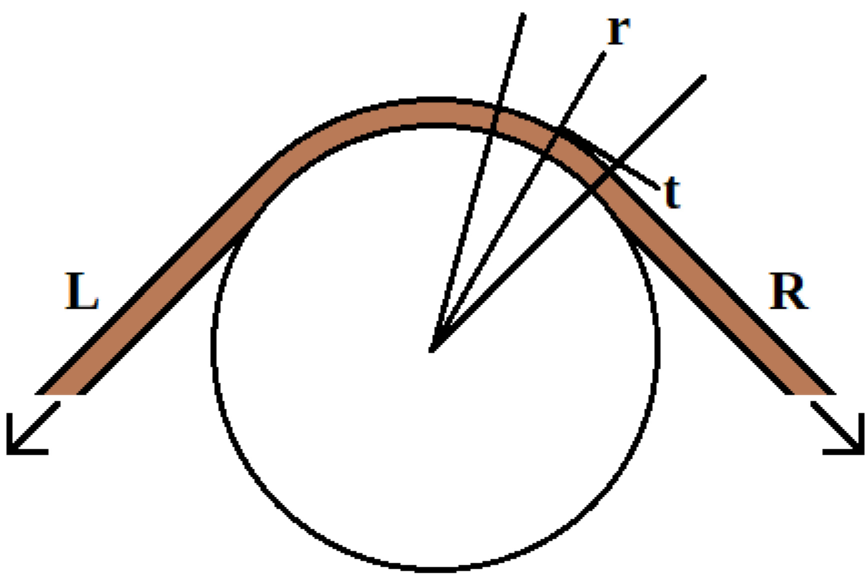

Figure 8: Rope around a pole.

Solution 1

In radial and tangential directions:

$$\sum F_r = 0 = N - F_L \sin\left(\frac{\theta}{2}\right) - F_R \sin\left(\frac{\theta}{2}\right)$$

$$\sum F_t = 0 = F_R \cos\left(\frac{\theta}{2}\right) - F_L \cos\left(\frac{\theta}{2}\right) - \mu N$$

Assume small contact angle, thus $F_L \approx F_R$:

$$0 \approx N - F\frac{d\theta}{2} - F\frac{d\theta}{2} \qquad\qquad 0 \approx dF - \mu N$$

Eliminate N and integrate:

$$0 \approx \frac{dF}{\mu} - F d\theta \qquad\qquad F = F_0\, e^{\mu\theta}$$

9.3 Circle Center from Three Points

Problem 1

Find the equation of a circle from three given points (x_j, y_j) where $j = 1, 2, 3$.

Solution 1

We must solve for h, k , and r in the system of three equations for the shifted circle

$$(x_j - h)^2 + (y_j - k)^2 = r^2 \qquad\qquad j = 1, 2, 3\,,$$

where h is the horizontal shift, k is the vertical shift, and r is the radius. Expanding the equation out for the three points given, we have:

$$\begin{aligned} x_1^2 + h^2 - 2x_1 h + y_1^2 + k^2 - 2y_1 k &= r^2 && \text{Equation (A)} \\ x_2^2 + h^2 - 2x_2 h + y_2^2 + k^2 - 2y_2 k &= r^2 && \text{Equation (B)} \\ x_3^2 + h^2 - 2x_3 h + y_3^2 + k^2 - 2y_3 k &= r^2 && \text{Equation (C)} \end{aligned}$$

Next, take the difference $(B) - (A)$, and also $(C) - (A)$ to get

$$\left(x_2^2 - x_1^2 + y_2^2 - y_1^2\right) + h\,(2x_1 - 2x_2) + k\,(2y_1 - 2y_2) = 0$$

$$\left(x_3^2 - x_1^2 + y_3^2 - y_1^2\right) + h\,(2x_1 - 2x_3) + k\,(2y_1 - 2y_3) = 0\,,$$

which is a system of linear equations in the variables h ad k. In tighter notation, the above equations read

$$a_1 + b_1 h + c_1 k = 0$$

$$a_2 + b_2 h + c_2 k = 0\,,$$

where a_i, b_i, c_i only depend on the known x_j, y_j. Solving the system for h and k separately, we finally discover:

$$h = \frac{a_1/b_1 - a_2/b_2}{c_2/b_2 - c_1/b_1} \qquad\qquad k = \frac{a_1/c_1 - a_2/c_2}{b_2/c_2 - b_1/c_1}$$

9.4 Circle Intersecting Circle

Problem 1

Find the intersection points of two overlapping circles having radii r_1, r_2 centered at $\vec{C}_1$, $\vec{C}_2$, respectively.

Solution 1

Let the parameterized vectors

$$\vec{r}_1 = r_1 \left\langle \cos\alpha, \sin\alpha \right\rangle \qquad\qquad \vec{r}_2 = r_2 \left\langle \cos\beta, \sin\beta \right\rangle$$

locate the position on each circle from their respective centers. The criteria for intersection is simply

$$\vec{C}_1 + \vec{r}_1 = \vec{C}_2 + \vec{r}_2 \ .$$

Letting

$$\vec{D} = \vec{C}_1 - \vec{C}_2$$

denote the displacement vector between the two centers, we have, in component form,

$$D_x = r_2 \cos\beta - r_1 \cos\alpha \qquad\qquad D_y = r_2 \sin\beta - r_1 \sin\alpha \ .$$

Solving for $\cos^2\alpha + \sin^2\alpha$, the above relations hand us

$$D_x \cos\beta + D_y \sin\beta = \frac{r_2^2 - r_1^2 + D_x^2 + D_y^2}{2r_2} = E \ ,$$

where E is a constant. Similarly solving for $\cos^2\beta + \sin^2\beta$, we get a similar form and a similar constant:

$$D_x \cos\alpha + D_y \sin\alpha = \frac{r_2^2 - r_1^2 - D_x^2 - D_y^2}{2r_1} = F$$

Defining a, b, c, d such that

$$a = \frac{D_x}{F} \qquad b = \frac{D_y}{F} \qquad c = \frac{D_x}{E} \qquad d = \frac{D_y}{E} \ ,$$

the whole problem is reduced to solving either of

$$a\cos\alpha + b\sin\alpha = 1 \qquad\qquad c\cos\beta + d\sin\beta = 1 \ .$$

Solving the first equation for $\cos\alpha$, we write

$$\begin{aligned} a\cos\alpha + b\sqrt{1-\cos^2\alpha} &= 1 \\ b^2\left(1-\cos^2\alpha\right) &= 1 - 2a\cos\alpha + a^2\cos^2\alpha \\ 0 &= \cos^2\alpha\left(a^2+b^2\right) + \cos\alpha\left(-2a\right) + \left(1-b^2\right) \ , \end{aligned}$$

where $\cos\alpha$ is readily given by the quadratic formula, simplifying to

$$\cos\alpha = \frac{a \pm b\sqrt{a^2+b^2-1}}{a^2+b^2} \ .$$

An identical exercise in solving for $\sin\alpha$ gives:

$$\sin\alpha = \frac{b \pm a\sqrt{a^2+b^2-1}}{a^2+b^2} \ .$$

Note that identical relations exist for the c, d, β equation by replacing

$$\alpha \to \beta \qquad a \to c \qquad b \to d \ ,$$

but this pursuit is redundant in the sense that α determines β, and vice versa.

Of course, quadratic solutions are multi-valued, and even worse, we're handed a pair of multi-valued solutions. This means the two $\pm$ symbols could be $++$, $+-$, $-+$, or $--$. In order to satisfy $a\cos\alpha + b\sin\alpha = 1$ however, only the mixed cases $+-$ and $-+$ remain valid. The pure $++$ and $--$ cases are tossed away. Denoting

$$g = a^2 + b^2 \qquad\qquad h = \sqrt{g-1}\,,$$

the solutions for α are:

$$\cos\alpha_1 = \frac{a \pm bh}{g} \qquad\qquad \sin\alpha_1 = \frac{b \mp ah}{g}$$

$$\cos\alpha_2 = \frac{a \mp bh}{g} \qquad\qquad \sin\alpha_2 = \frac{b \pm ah}{g}$$

Finally, the intersection points are given by

$$\vec{X}_{\text{int}} = \vec{C}_1 + r_1 \langle \cos\alpha_j, \sin\alpha_j \rangle \qquad\qquad j = 1, 2\,.$$

9.5 Co-Planar Points

Consider a three-dimensional space that embeds a triangle having vertex points

$$\vec{P}_j = \langle x_j, y_j, z_j \rangle \qquad\qquad (j = 1, 2, 3)\ ,$$

each measured from an origin at $\langle 0, 0, 0 \rangle$. Also embedded is a point

$$\vec{Q} = \langle Q_x, Q_y, Q_z \rangle$$

measured from the same origin as shown in the Figure.

Problem 1

Write an equation for the normal vector $\vec{N}$ that points perpendicular to the plane.

Problem 2

Express the location of $\vec{Q}$ in terms of the basis $\vec{U}$, $\vec{V}$, $\vec{N}$.

Problem 3

Write the condition for $\vec{Q}$ occurring in the same plane as the triangle.

Problem 4

For points $\vec{Q}$ co-planar to the triangle, determine the condition for $\vec{Q}$ being contained within the triangle.

Problem 5

Consider the triangle whose vertex points are

$$\vec{P}_1 = \langle 1, 0, 1 \rangle \qquad\qquad \vec{P}_2 = \langle 0, 1, 1 \rangle \qquad\qquad \vec{P}_3 = \langle 1, 1, 0 \rangle\ .$$

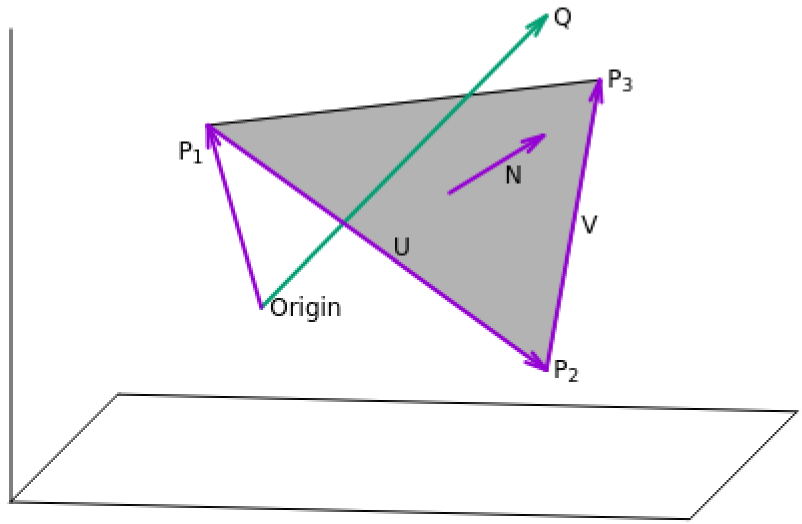

Figure 9: Triangle $P_1P_2P_3$ and point Q embedded in three dimensions.

Write an equation for all coplanar points $\vec{Q} = \langle x, y, z\rangle$ to the triangle.

Solution 1

Begin by defining two vectors $\vec{U}$, $\vec{V}$ in terms of the vertex points

$$\vec{U} = \vec{P}_2 - \vec{P}_1 \qquad\qquad \vec{V} = \vec{P}_3 - \vec{P}_2 \,.$$

Next, note that the cross product $\vec{U} \times \vec{V}$ yields precisely $\vec{N}$:

$$\vec{N} = \vec{U} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ U_x & U_y & U_z \\ V_x & V_y & V_z \end{vmatrix}$$

$$\vec{N} = (U_yV_z - V_yU_z)\,\hat{x} + (V_xU_z - U_xV_z)\,\hat{y} + (U_xV_y - V_xU_y)\,\hat{z}$$

Of course, we may always normalize the vector to have unit magnitude, meaning

$$\hat{n} = \frac{\vec{N}}{|N|} = \frac{\vec{U} \times \vec{V}}{\left|\vec{U} \times \vec{V}\right|} \qquad\qquad \hat{n} \cdot \hat{n} = 1 \,.$$

Solution 2

Tracing from the origin, we observe that

$$\vec{Q} = \vec{P}_1 + \alpha\,\vec{U} + \beta\,\vec{V} + \gamma\,\hat{n} \,,$$

where α, β, γ are dimensionless parameters. Interpreting this result, we may treat the triangle as a tilted xy-plane with $\vec{n}$ pointing in the 'vertical' direction, and the list $\langle \alpha, \beta, \gamma\rangle$ (the only unknowns in the problem) denotes a unique position in the three-dimensional space.

Solution 3

Project the unit normal vector $\hat{n}$ across the $\vec{Q}$-equation to get

$$\hat{n} \cdot \vec{Q} = \hat{n} \cdot \vec{P}_1 + \alpha \,\cancel{\hat{n} \cdot \vec{U}} + \beta \,\cancel{\hat{n} \cdot \vec{V}} + \gamma \, \hat{n} \cdot \hat{n} \,,$$

which can be solved for γ:

$$\gamma = \hat{n} \cdot \left(\vec{Q} - \vec{P}_1 \right)$$

Interpreting this result, γ measures the 'height' above (or below) the UV-plane. This means that all points in the plane of the triangle must have $\gamma = 0$:

$$0 = \hat{n} \cdot \left(\vec{Q} - \vec{P}_1 \right) \qquad\qquad \text{co-planarity condition}$$

Solution 4

So far we have two vectors $\vec{U}$, $\vec{V}$ along the sides of the triangle, but we may also denote the unlabeled side with another vector $\vec{W}$ such that

$$\vec{U} + \vec{V} + \vec{W} = 0 \,.$$

Next, observe that for a point to be inside the triangle, it must occur to the *left* of each vector $\vec{U}$, $\vec{V}$, $\vec{W}$. For a point to be left of $\vec{U}$, we must have

$$\hat{n} \cdot \vec{U} \times \left(\vec{Q} - \vec{P}_1 \right) = |U| \left| \vec{Q} - \vec{P}_1 \right| \sin \phi_1 > 0 \,,$$

where ϕ_1 is the angle formed between $\vec{U}$ and $\left(\vec{Q} - \vec{P}_1 \right)$. (If $\sin \phi_1$ turns out negative, then $\vec{Q}$ is to the *right* of $\vec{U}$, and is outside the triangle.) The same argument applies to the the $\vec{V}$ and $\vec{W}$-vectors, so the 'point-inside-triangle' condition contains two similar equations:

$$\hat{n} \cdot \vec{V} \times \left(\vec{Q} - \vec{P}_2 \right) = |V| \left| \vec{Q} - \vec{P}_2 \right| \sin \phi_2 > 0$$

$$\hat{n} \cdot \vec{W} \times \left(\vec{Q} - \vec{P}_3 \right) = |W| \left| \vec{Q} - \vec{P}_3 \right| \sin \phi_3 > 0$$

Solution 5

$$\vec{U} = \vec{P}_2 - \vec{P}_1 = \langle -1, 1, 0 \rangle \qquad\qquad \vec{V} = \vec{P}_3 - \vec{P}_2 = \langle 1, 0, -1 \rangle$$

$$\vec{N} = \vec{U} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle -1, -1, -1 \rangle \qquad\qquad \hat{n} = \frac{1}{\sqrt{3}} \langle -1, -1, -1 \rangle$$

$$0 = \hat{n} \cdot \left(\vec{Q} - \vec{P}_1 \right) = \frac{1}{\sqrt{3}} \langle -1, -1, -1 \rangle \cdot \langle x - 1, y - 0, z - 1 \rangle$$

$$0 = \frac{1}{\sqrt{3}} \left(-x + 1 - y - z + 1 \right)$$

$$x + y + z = 2$$

9.6 Cycloid

Problem 1

Let a ring of radius a roll without slipping along a flat surface. A fixed point P on the ring traces the shape of a *cycloid* as shown in the figure. Determine parametric equations for P in terms of the ring's rotation angle θ.

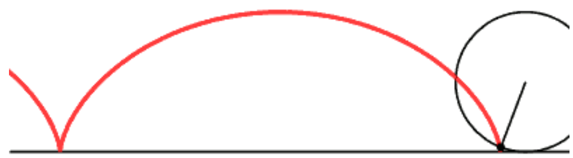

Figure 10: Generation of a cycloid. (*Wikipedia*)

Problem 2

Write down a pair of equations for the velocity of P as a function of time. Show that the associated velocity curve is a circle.

Problem 3 (Source: Calculus, Strang)

Show that the tangent line to the cycloid at any point P simultaneously passes through $x = a\theta$ and $y = 2a$.

Problem 4

Show that the normal line to the cycloid at any point P passes the point of contact of the generating circle with the straight line along which the circle rolls.

Problem 5 (Source: Calculus, Strang)

A stone in a bicycle tire goes along a cycloid. Find equations for the stone's path if it flies off the top.

Problem 6

Calculate the arc length along one iteration of a cycloid.

Problem 7

Calculate the area under one iteration of a cycloid.

Problem 8

Calculate volume thin one cycloid revolved around the line joining its endpoints.

Problem 9

Calculate surface area of one cycloid revolved around the line joining its endpoints.

Problem 10

A *tautochrone* is a curve for which a friction-less particle sliding downward due to gravity will reach the bottom at the same time T for all starting positions. Show that the cycloid satisfies this.

Problem 11

A *brachistochrone* is a curve for which a frictionless particle sliding downward due to gravity will its destination in a faster time T than by any other curve. Show that the cycloid satisfies this.

Solution 1

Placing the origin at the starting point of the cycloid, we must have

$$x\left(\theta\right) = a\theta - a\sin\theta \qquad\qquad y\left(\theta\right) = a - a\cos\theta\,.$$

Solution 2

Let ω be the time derivative of θ or $\dot{\theta}$. Then,

$$\dot{x} = a\omega - a\omega\cos\theta \qquad\qquad \dot{y} = a\omega\sin\theta\,,$$

leading to the equation of an offset circle

$$\left(\frac{\dot{x}}{\omega} - a\right)^2 + \left(\frac{\dot{y}}{\omega}\right)^2 = a^2\,.$$

Solution 3

For some special α, verify that

$$\left\langle a\theta, 2a\right\rangle = \left\langle x, y\right\rangle + \alpha\left\langle \dot{x}, \dot{y}\right\rangle\,.$$

Solution 4

The slope of the tangent normal line is $-1/\left(\dot{y}/\dot{x}\right)$. For some special β, verify that

$$\left\langle a\theta, 0\right\rangle = \left\langle x, y\right\rangle + \beta\left\langle -\dot{y}, \dot{x}\right\rangle\,.$$

Solution 5

$$x\left(t\right) = a\pi + 2a\omega t \qquad\qquad y\left(t\right) = 2a - gt^2/2$$

Solution 6

$$L = \int_0^{2\pi a}\sqrt{1 + \left(dy/dx\right)^2}\,dx = \sqrt{2}a\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta = 2a\int_0^{2\pi}\sin\left(\frac{\theta}{2}\right)\,d\theta = 8a$$

Solution 7

$$A = \int_0^{2\pi a} y\,dx = a^2\int_0^{2\pi}\left(1-\cos\theta\right)^2\,d\theta = 3\pi a^2$$

Solution 8

$$V = \int_0^{2\pi a} \pi y^2 \, dx = \pi a^3 \int_0^{2\pi} (1 - \cos\theta)^3 \, d\theta = \pi a^3 \cdot 5\pi = 5\pi^2 a^3$$

Solution 9

$$S = \int 2\pi y \, ds = 2\sqrt{2}\pi a^2 \int_0^{2\pi} (1 - \cos\theta)^{3/2} \, d\theta = 2\sqrt{2}\pi a^2 \cdot 16\sqrt{2}/3 = \frac{64\pi a^2}{3}$$

Solution 10

Use conservation of energy to write

$$T = \int \frac{ds}{v} = \int_{x_0}^{\pi a} \frac{\sqrt{1 + y'^2}}{\sqrt{2g\,(y_0 - y)}} \, dx = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{1 - \cos\theta}}{\sqrt{\cos\theta_0 - \cos\theta}} \, d\theta \,,$$

along with trig identities, and $u = (\cos\theta/2) \,/\, (\cos\theta_0/2)$ to seal the deal:

$$T = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sin(\theta/2) \; d\theta}{\sqrt{\cos^2\theta_0/2 - \cos^2\theta/2}} = \sqrt{\frac{a}{g}} \int_1^0 \frac{-2 \, du}{\sqrt{1 - u^2}} = \pi \sqrt{\frac{a}{g}}$$

Solution 11

Use conservation of energy to write

$$T = \int \frac{ds}{v} = \int_{y_0}^{0} \frac{\sqrt{1 + x'^2}}{\sqrt{2g\,(y_0 - y)}} \, dy = \int_{y_0}^{0} F(y, x') \, dy$$

where by method of variations we pick out $dF/dx' = A$, with A constant. Solve for dx/dy to arrive at the integral

$$x = \int \sqrt{\frac{(2gA^2)\,(y_0 - y)}{1 - (2gA^2)\,(y_0 - y)}} \, dy = -\int a\,(1 - \cos\theta) \, d\theta \,,$$

where the relations $a = 1/4gA^2$ and $y = y_0 - a\,(1 - \cos\theta)$ have been used. The minus sign in front of the x-integral arises because the inverted cycloid is generated by rolling backwards.

9.7 Ellipse Intersecting Ellipse

Problem 1

Find the intersection points of any ellipse crossing any other ellipse in two dimensions.

Problem 2

Consider two ellipses

$$\vec{r}_1 = a\cos\theta\,\hat{x} + b\sin\theta\,\hat{y}$$
$$\vec{r}_2 = b\cos\phi\,\hat{x} + a\sin\phi\,\hat{y} \,.$$

Determine the distance of any intersection point from the origin.

Solution 1

Each ellipse is parameterized in terms of an initial displacement vector $\vec{r}_0$ and the sum of two perpendicular terms that give the size and orientation of the ellipse

$$\vec{r}_1 = \vec{k}_1 + \vec{a}_1 \cos\theta + \vec{b}_1 \sin\theta$$
$$\vec{r}_2 = \vec{k}_2 + \vec{a}_2 \cos\phi + \vec{b}_2 \sin\phi \,,$$

where all vectors are in the same plane. Intersection points are simply where $\vec{r}_1 = \vec{r}_2$, so we must solve

$$\vec{k}_1 + \vec{a}_1 \cos\theta + \vec{b}_1 \sin\theta = \vec{k}_2 + \vec{a}_2 \cos\phi + \vec{b}_2 \sin\phi \,.$$

Multiplying by $\vec{a}_1$ and $\vec{b}_1$ respectively and denoting $\Delta\vec{k} = \vec{k}_2 - \vec{k}_1$, rearranging gives

$$\cos^2\theta = \left(\frac{\vec{a}_1 \cdot \Delta\vec{k} + \vec{a}_1 \cdot \vec{a}_2 \cos\phi + \vec{a}_1 \cdot \vec{b}_2 \sin\phi}{a_1^2}\right)^2$$
$$\sin^2\theta = \left(\frac{\vec{b}_1 \cdot \Delta\vec{k} + \vec{b}_1 \cdot \vec{a}_2 \cos\phi + \vec{b}_1 \cdot \vec{b}_2 \sin\phi}{b_1^2}\right)^2 \,,$$

where using the identity $\cos^2\theta + \sin^2\theta = 1$, we reduce the problem to solving

$$1 = \left(\frac{\vec{a}_1 \cdot \Delta\vec{k} + \vec{a}_1 \cdot \vec{a}_2 \cos\phi + \vec{a}_1 \cdot \vec{b}_2 \sin\phi}{a_1^2}\right)^2 + \left(\frac{\vec{b}_1 \cdot \Delta\vec{k} + \vec{b}_1 \cdot \vec{a}_2 \cos\phi + \vec{b}_1 \cdot \vec{b}_2 \sin\phi}{b_1^2}\right)^2$$

for ϕ. It turns out this equation is highly transcendental, and can be solved in the general case numerically if needed.

Solution 2

Begin with the above formula and set $\Delta\vec{k} = 0$ as given, along with

$$\vec{a}_1 = a\,\hat{x} \qquad \vec{b}_1 = b\,\hat{y} \qquad \vec{a}_2 = b\,\hat{x} \qquad \vec{b}_2 = a\,\hat{y} \,,$$

reducing the above to

$$1 = \frac{b^2}{a^2}\cos^2\phi + \frac{a^2}{b^2}\sin^2\phi \,,$$

telling us

$$\cos\phi = \frac{\pm a}{\sqrt{a^2+b^2}} \qquad \sin\phi = \frac{\pm b}{\sqrt{a^2+b^2}} \,.$$

Taking the positive roots, one intersection point of the ellipses is given by

$$\vec{r}^{*} = b\cos\phi\,\hat{x} + a\sin\phi\,\hat{y} = \frac{ab}{\sqrt{a^2+b^2}}\,\hat{x} + \frac{ab}{\sqrt{a^2+b^2}}\,\hat{y}$$
$$= \frac{ab}{\sqrt{a^2+b^2}}\left(\hat{x}+\hat{y}\right) \,.$$

The distance from any intersection point to the origin is evidently (as shown in the Figure)

$$|\vec{r}^{*}| = \frac{\sqrt{2}ab}{\sqrt{a^2+b^2}} \,.$$

Note that in the limit $a \to \infty$, the shape enclosed becomes a square of side length b.

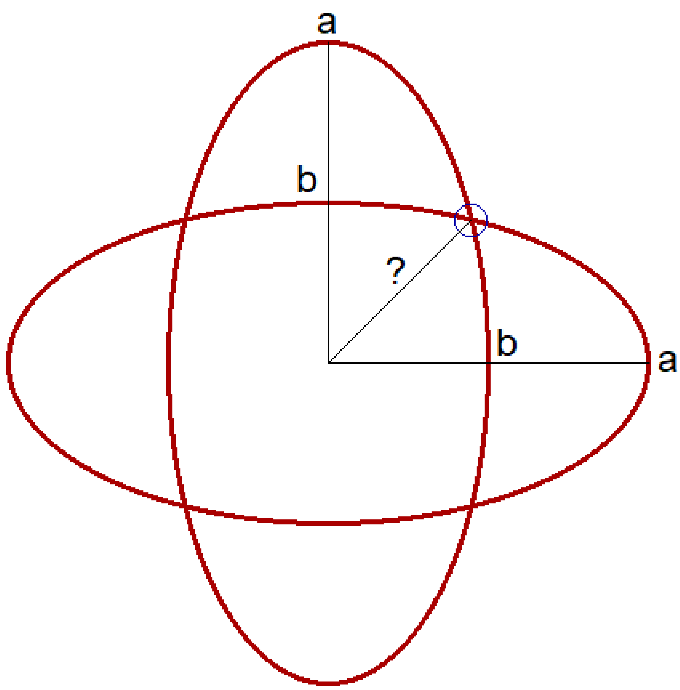

Figure 11: Special case of intersecting ellipses.

9.8 Ellipse Intersecting Parabola

Problem 1

Find the intersection points of any ellipse crossing any parabola in two dimensions.

Problem 2

Consider two curves

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$
$$y = Ax^2 + Bx + C \; .$$

Write an equation that can be used to calculate the intersection points.

Solution 1

An ellipse is parameterized in terms of an initial displacement vector $\vec{r}_0$ and the sum of two perpendicular terms that give the size and orientation of the ellipse

$$\vec{r}_e = \vec{r}_0 + \vec{a}\cos\theta + \vec{b}\sin\theta \; ,$$

where $\vec{a}$ and $\vec{b}$ live in the $x4$-plane. Any parabola in the xy-plane is represented by

$$\vec{y}_p = x\,\hat{x} + y\,(x)\;\hat{y} \; ,$$

where $y\,(x) = Ax^2 + Bx + C$.

Intersections between the two curves occur when $\vec{r}_e = \vec{r}_p$, namely

$$\vec{r}_0 + \vec{a}\cos\theta + \vec{b}\sin\theta = x\,\hat{x} + y\,\hat{y}\,.$$

To proceed, multiply through by $\hat{a}$ and $\hat{b}$ respectively to get two equations

$$\begin{aligned}\vec{a}\cdot\vec{r}_0 + a^2\cos\theta &= \vec{a}\cdot(\vec{x}+\vec{y}\,(x))\\ \vec{b}\cdot\vec{r}_0 + b^2\sin\theta &= \vec{b}\cdot(\vec{x}+\vec{y}\,(x))\ ,\end{aligned}$$

where θ can be eliminated using $\cos^2\theta + \sin^2\theta = 1$ to give

$$\left(\frac{-\vec{a}\cdot\vec{r}_0 + \vec{a}\cdot(\vec{x}+\vec{y}\,(x))}{a^2}\right)^2 + \left(\frac{-\vec{b}\cdot\vec{r}_0 + \vec{b}\cdot(\vec{x}+\vec{y}\,(x))}{b^2}\right)^2 = 1\,,$$

which determines the general case, as x is the only remaining unknown.

Solution 2

Begin with the above formula and set $\vec{r}_0 = 0$ as given, along with

$$\vec{a} = a\,\hat{x} \qquad\qquad \vec{b} = b\,\hat{y}$$

to get

$$\begin{aligned}\left(\frac{\hat{x}\cdot(\vec{x}+\vec{y}\,(x))}{a}\right)^2 + \left(\frac{\hat{y}\cdot(\vec{x}+\vec{y}\,(x))}{b}\right)^2 &= 1\\ \left(\frac{x}{a}\right)^2 + \left(\frac{Ax^2+Bx+C}{b}\right)^2 &= 1\,,\end{aligned}$$

which is a determined equation of a single variable.

9.9 Equatorial Belt

Source: Nick's Mathematical Puzzles
`http://www.qbyte.org/puzzles/puzzle01.html`

Problem 1

A snug-fitting belt is placed around the Earth's equator. Suppose you added an extra 1 meter of length to the belt, held it at a point, and lifted until all the slack was gone. How high above the Earth's surface would you then be? That is, find h in the diagram.

Solution 1

Observe

$$\frac{1\,m}{2r} = \frac{a}{r} - x \qquad\qquad \tan x = \frac{a}{r}\,,$$

where a/r can be eliminated by writing at $1\,m/2r = \tan x - x$. Since the radius of the Earth is much greater than one meter, we see that $x \approx a/r$, and also $\tan \approx x$, so x too is small. Using a series expansion for $\tan x$, we have $1\,m/2r \approx x^3/3$.

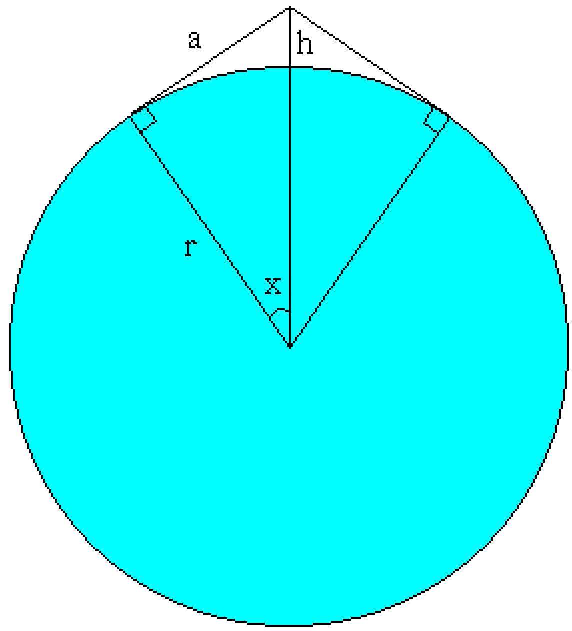

Figure 12: Equatorial Belt.

Finally, using $(r + h) \cos x = r$, we may write

$$(6400\,km + h) \cos \left[\left(\frac{3\,m}{6400\,km} \right)^{1/3} \right] = 6400\,km \ ,$$

which resolves to $h \approx 121.6\,m$.

9.10 Inscribed Circle in Right Triangle

Source: Stewart Calculus
`www.stewartcalculus.com/data/default/upfiles/7e_challprobs_student.pdf`

Problem 1

Let ABC be a triangle with right angle A and hypotenuse $|BC|$. (See the figure.) If the inscribed circle of radius R touches the hypotenuse at D, show that:

$$|CD| = \frac{1}{2} \left(|BC| + |AC| - |AB| \right)$$

Problem 2

Show that the right triangle obeying $2\,|CD| = |AB|$ is a $3, 4, 5$ triangle.

Problem 3

Derive a formula for $|AC| + |CD|$ in terms of R and angle ABC.

Solution 1

Begin by setting the origin at A, and also define four unit vectors:

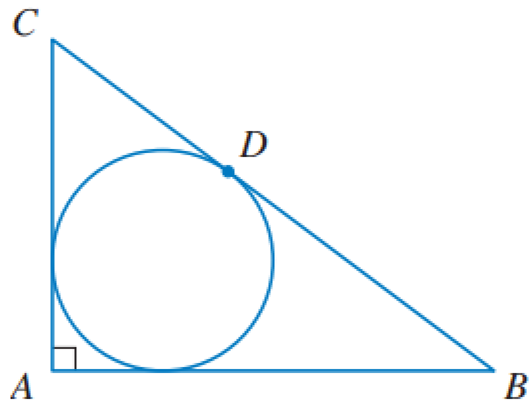

Figure 13: A circle embedded in a right triangle.

- Let $\hat{x}$ be aligned with $|AB|$.
- Let $\hat{y}$ be aligned with $|AC|$.
- Let $\hat{h}$ be aligned with $|CB|$.
- Let $\hat{h}'$ be perpendicular to $\hat{h}$, pointing away from the triangle.

By vector addition, there two obvious ways to arrive at point D, starting from A:

$$|AC|\,\hat{y} + |CD|\,\hat{h} = |AB|\,\hat{x} - |BD|\,\hat{h}$$

There is a third way to reach point D however, which involves stepping to the center of the circle, and then once more to reach D. Calling the circle radius R, and then using the notation on hand, we find:

$$\text{vector AD} = R\hat{x} + R\hat{y} + R\hat{h}'$$

Combining our findings into one relation, arrive at

$$R\left(\hat{x} + \hat{y} + \hat{h}'\right) = |AC|\,\hat{y} + |CD|\,\hat{h} = |AB|\,\hat{x} - |BD|\,\hat{h}\,,$$

embodying the problem statement along with our insights into it. Proceed by taking projections of the above equation in the x-, y-, h-, and h'-directions. Before getting dirty, note that our choice of unit vectors obey the following relations:

$$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = 1 \qquad \hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{x} = 0$$

$$\hat{h}\cdot\hat{h} = \hat{h}'\cdot\hat{h}' = 1 \qquad \hat{h}\cdot\hat{h}' = \hat{h}'\cdot\hat{h} = 0$$

Note also that $\hat{h}'$ is a ninety-degree rotation of $\hat{h}$, so we also have

$$\hat{h}'\cdot\hat{x} = -\hat{h}\cdot y \qquad \hat{h}'\cdot\hat{y} = \hat{h}\cdot\hat{x}\,.$$

Next we project $\hat{x}$, $\hat{y}$, $\hat{h}$, and $\hat{h}'$, into the main equation, giving four results. Note that the dot product relations above are used to eliminate all references to $\hat{h}'$.

$$R\left(1-\hat{h}\cdot\hat{y}\right) = |CD|\,\hat{h}\cdot\hat{x} = |AB| - |BD|\,\hat{h}\cdot\hat{x}$$

$$R\left(1+\hat{h}\cdot\hat{x}\right) = |AC| + |CD|\,\hat{h}\cdot\hat{y} = -\,|BD|\,\hat{h}\cdot\hat{y}$$

$$R\left(\hat{h}\cdot\hat{x}+\hat{h}\cdot\hat{y}\right) = |AC|\,\hat{h}\cdot\hat{y} + |CD| = |AB|\,\hat{h}\cdot\hat{x} - |BD|$$

$$R\left(-\hat{h}\cdot\hat{y}+\hat{h}\cdot\hat{x}+1\right) = |AC|\,\hat{h}\cdot\hat{x} = -\,|AB|\,\hat{h}\cdot\hat{y}$$

Adding the first three equations and then subtracting the last, we find, after combining like terms,

$$R\left(1+\hat{h}\cdot\hat{x}+\hat{h}\cdot\hat{y}\right) = \left(|AB| - |BD|\right)\left(1+\hat{h}\cdot\hat{x}+\hat{h}\cdot\hat{y}\right) \; ,$$

allowing the factor with the dot products to be canceled

Evidently, we have derived

$$R = |AB| - |BD| \; .$$

Note this result could have been picked out from the get-go, which is reassuring, as our pedantic formalism *should* indicate all intuitively-obvious features of the problem. Moving on, note that by symmetry/choice of coordinate system we may effectively swap the labels B and C and the problem retains its structure. We use this to blurt out the sister relation to the above:

$$R = |AC| - |CD| \; .$$

Eliminating the radius R gives

$$|AB| - |BD| = |AC| - |CD| \; ,$$

and using $|BD| = |BC| - |CD|$ gives

$$|AB| - |BC| + |CD| = |AC| - |CD| \; .$$

Solving for $|CD|$, the desired result emerges:

$$|CD| = \frac{1}{2}\left(|BC| + |AC| - |AB|\right)$$

Solution 2

Replacing $2\,|CD|$ with $|AB|$ and using the Pythagorean theorem to eliminate $|BC|$, the main equation reduces to $3\,|AB| = 4\,|AC|$.

Solution 3

Let θ equal angle ABC. It follows that

$$\hat{h}\cdot\hat{x} = \cos\theta \qquad\qquad \hat{h}\cdot\hat{y} = -\sin\theta \; .$$

Next add the four sequentially-listed equations above and combine like terms:

$$R\left(3+3\hat{h}\cdot\hat{x}-\hat{h}\cdot\hat{y}\right) = \left(|AC| + |CD|\right)\left(1+\hat{h}\cdot\hat{x}+\hat{h}\cdot\hat{y}\right)$$

Simplify to arrive at:

$$R\left(\frac{3+3\cos\theta-\sin\theta}{1+\cos\theta-\sin\theta}\right) = |AC| + |CD|$$

9.11 Inscribed Rectangle in Right Triangle

Problem 1

Identify the largest rectangle that fits inside a $3, 4, 5$ right triangle where one of the rectangle's edges lies on the hypotenuse.

Solution 1

Define θ such that $5 \cos\theta = 4$ and $5 \sin\theta = 3$. The side of the rectangle parallel to the hypotenuse has length $\sqrt{(3-x)^2 + (4-y)^2}$, and the second rectangle dimension is given by $x \cos\theta = y \sin\theta$. Proceed by either maximizing

$$A_1 = x \cos\theta \sqrt{(3-x)^2 + (4-y)^2}\,,$$

or by minimizing

$$A_2 = \frac{1}{2}x^2 \cos\theta \sin\theta + \frac{1}{2}y^2 \cos\theta \sin\theta + \frac{1}{2}(3-x)(4-y)$$

to get $x = 3/2$. (Note that $A_1 + A_2 = 6$.)

9.12 Inscribed Triangle in Square

Source: Brilliant
`https://brilliant.org/`

Problem 1

Determine the unknown blue area contained in the square as shown.

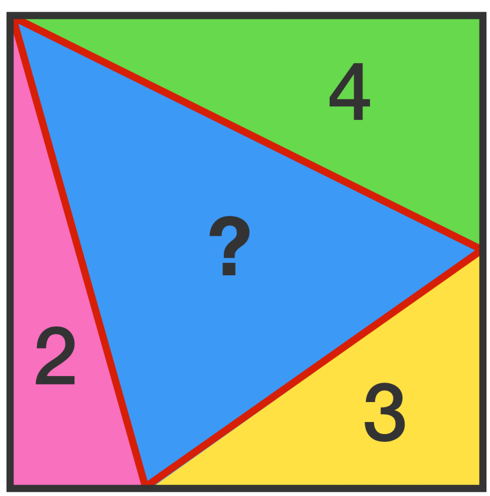

Figure 14: A triangle embedded in a square.

Solution 1

Starting on the left side and reading counter-clockwise, the four perimeter segments are (A), $(B+C)$, $(D+E)$, (A). Note also that

$$AB = 4 \qquad CD = 6 \qquad AE = 8\,.$$

Use $(A-B)(A-E) = CD$ to arrive at

$$A^2 + \frac{32}{A^2} = 18\,,$$

which resolves to $A = 4$, making the unknown area equal to 7.

9.13 Ladder Touching Cube

Source: stirlingsouth.com
`http://www.stirlingsouth.com/richard/trig9.htm`

Problem 1

A 20-foot ladder leans on a perpendicular wall such that it touches the edge of a $6\times6\times6$-foot cube flatly pushed against the wall, as seen in the figure. Find the vertical height of the ladder above the cube. Bonus: solve the problem analytically without writing down a fourth-order polynomial.

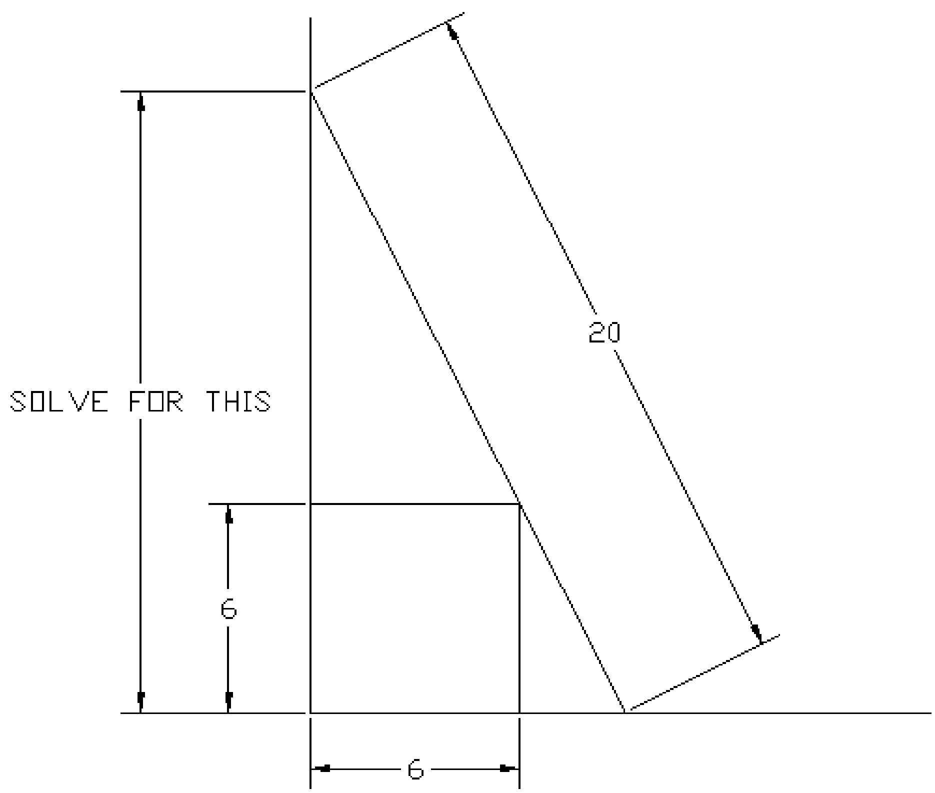

Figure 15: Ladder touching cube.

Solution 1

Denote x as the unknown vertical component of the ladder's projection, and y as the unknown horizontal component. By area arguments, or by considering similar triangles, observe that $xy = 36$. Further, the Pythagorean theorem dictates

$$(x+6)^2 + (y+6)^2 = 20^2 ,$$

where completing the square in the variable $x + y$ eventually gives

$$x + y = -6 + 2\sqrt{109} ,$$

resolving to $x \approx 11.840$, $y \approx 3.0405$.

9.14 Line Segments Intersecting

Consider a line segment limited by two points $\vec{p}_A$, $\vec{p}_B$, along with a second line segment in the same plane limited by $\vec{q}_A$, $\vec{q}_B$.

Problem 1

Determine if and where the line segments intersect.

Problem 2

Explore the case of parallel and overlapping line segments.

Solution 1

Each line segment can be expressed by a parameterized equation $\vec{y}_j$ such that

$$\vec{y}_j = \vec{b}_j + \alpha_j \, \hat{t}_j ,$$

where $\vec{b}_j$ is any point on the line (perhaps the y-intercept), $\hat{t}_j$ is a unit vector tangent to the line, and the dimensionless parameter α tracks valid positions on the line segment. The α-parameter is valid in a window such that

$$\alpha_{\text{min}} \leq \alpha \leq \alpha_{\text{max}} ,$$

where α_{min}, α_{max} coincide with the endpoints of the line segment. (Note that either limit can be extended to $\pm\infty$ to handle semi-infinite rays and infinite lines.)

For two line segments p and q, the intersection point (if any) is indicated by $\vec{y}_p = \vec{y}_q$, or

$$\vec{b}_p + \alpha_p \, \hat{t}_p = \vec{b}_q + \alpha_q \, \hat{t}_q .$$

Denoting

$$\Delta\vec{b} = \vec{b}_p - \vec{b}_q \qquad f = \hat{t}_p \cdot \hat{t}_q \qquad w_q = \Delta\vec{b} \cdot \hat{t}_q \qquad w_p = \Delta\vec{b} \cdot \hat{t}_p ,$$

the above can generate two equations

$$\alpha_q - \alpha_p f = w_q \qquad \alpha_q f - \alpha_p = w_p .$$

Solving for each α-parameter, we find the pair of results

$$\alpha_q = \frac{w_q - f w_p}{1 - f^2} \qquad\qquad \alpha_p = \frac{f w_q - w_p}{1 - f^2} \; .$$

Of course, the above solutions calculate intersections of non-parallel infinite lines. To restrict valid solutions to rays or segments, each $\alpha_{q,p}$ should satisfy

$$\alpha_{q,p_{\min}} \leq \alpha_{q,p} \leq \alpha_{q,p_{\max}} \; .$$

Solution 2

Parallel lines have the same tangent vector, causing divergence in each α-solution due to $f = \hat{t}_p \cdot \hat{t}_q = 1$. In the general case, parallel lines don't intersect, however overlapping parallel lines, or also potentially overlapping line segments, must exhibit the property

$$\Delta\vec{b} = \Delta b \, \hat{t}_{p \text{ or } q} \; .$$

That is, the vector $\Delta\vec{b}$ is along the overlapping lines.

Using the above notation, we can solve for the position(s) on the p-line segment that coincide(s) with any endpoint(s) on the q-line, and vice versa, giving

$$\alpha_q = \alpha_{p_{\text{crit}}} \, \hat{t}_q \cdot \hat{t}_p + \hat{t}_q \cdot \Delta\vec{b} \qquad\qquad \alpha_p = \alpha_{q_{\text{crit}}} \, \hat{t}_p \cdot \hat{t}_q - \hat{t}_p \cdot \Delta\vec{b} \, ,$$

where $\alpha_{p_{\text{crit}}}$ represents each of $\alpha_{p_{\min}}$, $\alpha_{p_{\max}}$, and similarly for $\alpha_{q_{\text{crit}}}$. As found previously, valid solutions $\alpha_{q,p}$ should satisfy

$$\alpha_{q,p_{\min}} \leq \alpha_{q,p} \leq \alpha_{q,p_{\max}} \; .$$

9.15 Line Intersecting Ellipse

Problem 1

Find the intersection points of any straight line crossing any ellipse in two dimensions.

Solution 1

A line in space is parameterized in terms of an initial displacement $\vec{b}_0$ from the origin, plus a parameter α multiplied by a unit vector $\hat{v}$ that tracks the direction:

$$\vec{r}_l = \vec{b}_0 + \alpha \, \hat{v}$$

An ellipse is parameterized in terms of an initial displacement vector $\vec{r}_0$ and the sum of two perpendicular terms that give the size and orientation of the ellipse

$$\vec{r}_e = \vec{r}_0 + \vec{a} \cos\theta + \vec{b} \sin\theta \; ,$$

where vectors $\vec{a}$ and $\vec{b}$ are in the same plane as $\hat{v}$.

The intersection of the line and the ellipse is simply given by $\vec{r}_l = \vec{r}_e$, as in

$$\vec{b}_0 + \alpha \, \hat{v} = \vec{r}_0 + \vec{a} \cos\theta + \vec{b} \sin\theta \; ,$$

which ought to have at most two solutions. To proceed, take two copies of the above equation and multiply through by $\vec{a}$ and $\vec{b}$, respectively to get:

$$\vec{a} \cdot \vec{b}_0 + \alpha\, \vec{a} \cdot \hat{v} = \vec{a} \cdot \vec{r}_0 + a^2 \cos\theta$$

$$\vec{b} \cdot \vec{b}_0 + \alpha\, \vec{b} \cdot \hat{v} = \vec{b} \cdot \vec{r}_0 + b^2 \sin\theta$$

While these look like very busy equations, note that everything is known except for α and θ. To proceed, exploit a fundamental trig identity to get

$$\cos^2\theta + \sin^2\theta = 1 = \left(\frac{\vec{a} \cdot \left(\vec{b}_0 - \vec{r}_0\right) + \alpha\, \vec{a} \cdot \hat{v}}{a^2} \right)^2 + \left(\frac{\vec{b} \cdot \left(\vec{b}_0 - \vec{r}_0\right) + \alpha\, \vec{b} \cdot \hat{v}}{b^2} \right)^2 ,$$

where α is the only unknown. Blooming out the algebra, we have

$$0 = A\alpha^2 + B\alpha + C ,$$

where

$$A = \left(\frac{\vec{a} \cdot \hat{v}}{a^2}\right)^2 + \left(\frac{\vec{b} \cdot \hat{v}}{b^2}\right)^2$$

$$B = \frac{2\vec{a} \cdot \left(\vec{b}_0 - \vec{r}_0\right)(\vec{a} \cdot \hat{v})}{a^4} + \frac{2\vec{b} \cdot \left(\vec{b}_0 - \vec{r}_0\right)\left(\vec{b} \cdot \hat{v}\right)}{b^4}$$

$$C = \left(\frac{\vec{a} \cdot \left(\vec{b}_0 - \vec{r}_0\right)}{a^2} \right)^2 + \left(\frac{\vec{b} \cdot \left(\vec{b}_0 - \vec{r}_0\right)}{b^2} \right)^2 - 1 .$$

Finally, the two solutions for α emerge from the quadratic formula:

$$\alpha = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

In the special case $B^2 = 4AC$, the one solution for α corresponds to the tangent line. If the radical becomes imaginary, the line misses the ellipse.

9.16 Railroad-Rail Classic

Source: Holyoke Community College, Calculus 3

Problem 1

A one-mile rail, fixed at its ends, is lengthened by one foot so that it bows up in a circular arc. Find the maximum height, d, without losing 4 or 5 digits in your computation.

Solution 1

Let the circular arc have radius R and be swept out by angle 2θ. Pick out three relations

$$R \sin\theta = \frac{5280}{2} \qquad R2\theta = 5281 \qquad d = R\left(1 - \cos\theta\right) ,$$

which resolves to $d \approx 44.4985\ ft$.