Measure and Number

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1 Units and Dimensional Analysis

Fundamental Dimensions

All physical quantities carry some kind of *dimensionality*, which is a way of tying a number to the physical world. There are only three* fundamental physical dimensions, which are (including abbreviations):

Length (L) Mass (M) Time (T)

The only quantities that need not be expressed in terms of fundamental dimensions are pure mathematical numbers, such as 3 or π.

* It should be pointed out that 'dimension' in this context does not refer to the idea that space has three dimensions (length, width, depth).

Systems of Units

History has handed down (at least) three different unit systems for keeping track of dimensionality.

SI Systems

The abbreviation *SI* stands for Système International, commonly known as the *metric system*. There are two 'factions' within the metric system: one is called *MKS* and the other is called *CGS*. The letters *M-K-S* tell us to use Meters, Kilograms, and Seconds, respectively, as fundamental units. Meanwhile, the *C-G-S* prefers Centimeters, Grams, and Seconds. That is:

System	**Length**	**Mass**	**Time**
MKS	Meter (m)	Kilogram (kg)	Second (s)
CGS	Centimeter (cm)	Gram (g)	Second (s)

English System

Only three countries besides Great Britain have so far resisted the metric system's campaign to dominate all standards: Burma, Liberia, and the United States. The *English system* offers another set of labels for the fundamental dimensions, those being the Foot, the Slug, and the Second. (Despite its popularity, the English pound is not fundamental.) We add to our table of fundamental units:

System	**Length**	**Mass**	**Time**
English	Foot (ft)	Slug (rarely used)	Second (s)

Expensive Mistake by NASA

Forgetting to properly handle units can be an expensive mistake, namely $328 million when NASA lost the Mars Climate Orbiter. In September 1999, NASA attempted to put a satellite close enough to Mars to study its atmosphere and surface features. However, a ground-based computer in communication with the Orbiter was sending information using the *English* system of units! Specifically, the Orbiter expected a packet of information expressed in Newtons, but it was interpreted as Pounds. The Orbiter didn't steer correctly and slammed into Mars.

Standard Units

As of yet, there is no pure system of units implied by nature alone - there is always a human element involved in setting standards.

Standard of Length

Length standards can be traced back to ancient history, with one popular unit of measurement being the the elbow-to-fingertip distance on a human arm in an L-shape, known as the 'cubit'. Obviously, the cubit varies form person to person, and more drastically between separate populations, and thus was not at all standard.

From Europe we gained another measure of length, the familiar 'foot', which was once defined by the length of the ruling king's actual foot. It wasn't until the days of Louis XIV that the foot was standardized in France. The name was upgraded to the 'Royal Foot'.

Around 1795, the standard length unit was decidedly the Meter, and was defined as 1/20000000 the length of Earth's prime meridian (this was before orthodox geology was settled on).

By the late 1800's, the standard length unit was determined by the length of one particular (carefully protected) platinum-iridium bar, whose length was simply declared to equal precisely one meter. Precise copies of this bar were produced and shipped off to spread the length standard around the world.

Eventually, experimental physics would demand a more precise definition for the length standard that was robust with respect to special relativity, which inevitably involves the speed of light. *The meter is precisely defined as the distance light travels in* $1/299792458$ *sec in vacuum.*

Standard of Mass

The modern definition of the Kilogram is as new as 2019. Before a (relatively) recent decision to uproot tradition, the *kilogramme* was an *actual* chunk of metal (90% platinum, 10% iridium) whose mass was decidedly one kilogram by popular vote. It's likely that the calibration of all science-grade mass scales and standards of the time could be traced back to the *kilogramme*, housed in France for its career.

There was curious problem with the *kilogramme* though: it's mass has slowly *increased* over history, and much effort has been devoted to finding out why this appears so. Wth or without a perfect explanation for this, the conclusion we must draw is that a chunk of metal is ultimately *not* a good mass standard.

The General Conference on Weights and Measures (CGPM) tried repeatedly to settle a definition of the kilogram that arises from quantum mechanics, specifically involving the Planck mass. They couldn't decide in 2011, and procrastinated again in 2014. Eventually, laboratory measurements of Planck's constant became sufficiently refined (about 13 ppb uncertainty) such that the definition of one kilogram is simply back-read out of the result.

Standard of Time

One reasonable definition for the Second is to slice a 24-hour interval into $60 \cdot 60 \cdot 24$ parts (like an analog watch). This was considered reliable until 1967, until it was decided that the fundamental unit of time had better take on a less 'Earthly' definition. It was then decided that one second would be defined as precisely 9192631770 periods of the light waves emitted from cesium decay.

There was nothing special about cesium except it was an available atom for study. There's nothing terribly special about the number 9192631770, except it seemed good enough for reproducing the 'old' value of one second. Going through the trouble to define the second this way allows scientists to deal more easily with many practical issues, such as adding leap-seconds to the calendar, and calibrating clocks far from Earth. (The latter was on their minds in the 1960's more so than it is today.)

Combinations of Units

There are many combinations of units - some are familiar, some are rarely used.

Combination	**Label**	**MKS**	**SI**	**English**
L^2	Area	m^2	$(cm)^2$	$(ft)^2$
L^3	Volume	m^2	$(cm)^3$	$(ft)^3$
L/T	Speed	m/s	cm/s	ft/s
L/T^2	Acceleration	m/s^2	cm/s^2	ft/s^2
$M \cdot L/T$	Momentum	$kg \cdot m/s$	$g \cdot cm/s$	(rare)
$M \cdot L/T^2$	Force	$kg \cdot m/s^2 = N$	$g \cdot cm/s^2$	Pound
M/L^3	Mass Density	kg/m^3	g/cm^3	(rare)
$M/\left(LT^2\right)$	Pressure	$N/m^2 = Pa$	$.1\ Pa$	PSI
$M \cdot L^2/T^2$	Energy	$N \cdot m = J$	$g \cdot cm^2/s^2$	Calorie

Dimensional Analysis

Quantities of the same fundamental dimensionality can be converted from one system of units to another using *dimensional analysis*. One may convert inches to centimeters, or kilometers to miles, or liters to gallons because these conversions preserve the dimension of the quantity. On the other hand, trying to convert microseconds to pounds is a nonsensical attempt to equate a time to a weight.

Conversion Table

Following are several common unit equivalences:

$$\begin{aligned}
1\text{ inch} &= 2.54\text{ centimeter}\\
1\text{ foot} &= 12\text{ inch} = (1/3)\text{ yard}\\
1\text{ mile} &= 5280\text{ foot} = 1.609\text{ kilometer}\\
1\text{ meter} &= 1.094\text{ yard}\\
1\text{ gallon} &= 128\text{ fluid ounce} = 3.785\text{ liter}\\
1\text{ hour} &= 60\text{ minute} = 3600\text{ second}\\
1\ cm^3 &= 3.785\text{ milliliter}\\
1\ kg &= 2.2\text{ pound (at sea level)}\\
360^\circ &= 2\pi\text{ radian}
\end{aligned}$$

Unit Conversion

Consider the maneuver of converting 5 inches to the equivalent in centimeters:

$$5\ in = 5\ \cancel{in}\left(\frac{2.54\ cm}{1\ \cancel{in}}\right) = 12.7\ cm$$

One interpretation of what just occurred is 5 inches was multiplied by a carefully-chosen factor of one. The same operation also works in reverse by inverting the factor of one:

$$12.7\ cm = 12.7\ \cancel{cm}\left(\frac{1\ in}{2.54\ \cancel{cm}}\right) = 5\ in$$

Areas and Volumes

Of course, 1^2 is still 1, so we multiply by 1^2 or 1^3, etc., for converting areas, volumes, and so on. For instance:

$$25\ in^2 = 25\ \cancel{in^2}\left(\frac{2.54^2\ cm^2}{1^2\ \cancel{in^2}}\right) = 25\cdot 2.54^2\ cm^2 \approx 161\ cm^2$$

Example

Convert 70 meters per second to milers per hour.

$$\begin{aligned}
70\ m/s =& 70\left(\frac{\cancel{m}}{\cancel{s}}\right)\left(\frac{1\ \cancel{km}}{1000\ \cancel{m}}\right)\left(\frac{1\ mi}{1.609\ \cancel{km}}\right)\left(\frac{3600\ \cancel{s}}{1\ hr}\right)\\
=& \frac{70\times 3600}{1000\times 1.609}\left(\frac{mi}{hr}\right) = 156.6\ mph
\end{aligned}$$

Example

Convert 4500 gallons to cubic meters.

$$
\begin{aligned}
4500\,gal =& 4500\,\cancel{gal}\left(\frac{3.785\,\cancel{liter}}{1\,\cancel{gal}}\right)\left(\frac{1000\,\cancel{mL}}{1\,\cancel{liter}}\right)\left(\frac{1\,\cancel{cm^3}}{1\,\cancel{mL}}\right)\left(\frac{1\,m}{100\,\cancel{cm}}\right)^3 \\
=& \frac{4500 \times 3.785 \times 1000}{100^3}\,m^3 = 17.03\,m^3
\end{aligned}
$$

Example

Convert the Massachusetts highway speed limit of 65 *mph* into lightyears per second. (A lightyear is a *distance*, equal to the distance that a beam of light will travel in one year.) Use the information below:

$$
\begin{aligned}
c = \text{speed of light} &= 3.00 \times 10^8\,m/s \\
1\text{ lightyear} &= c\ \times 365.25\text{ day}
\end{aligned}
$$

$$
\begin{aligned}
65\,mph =& 65\left(\frac{\cancel{mi}}{\cancel{hr}}\right)\left(\frac{\cancel{hr}}{3600\,\cancel{sec}}\right)\left(\frac{1609\,\cancel{m}}{1\,\cancel{mi}}\right) \\
&\times\left(\frac{c\,\cancel{sec}}{3.00\times 10^8\,\cancel{m}}\right)\left(\frac{365.25\,day}{365.25\,\cancel{day}}\right)\left(\frac{\cancel{day}}{24\times 3600\,sec}\right) \\
&= \frac{65 \times 1609}{3600^2 \times 3.00 \times 10^8 \times 365.25}\,ly/s \\
&= 3.07 \times 10^{-15}\,ly/s
\end{aligned}
$$

2 Scientific Notation

Order of Magnitude

Every quantity, regardless of its dimensionality, has an *order of magnitude*, which tells us how many powers of ten the quantity contains. For instance, the number four thousand is written 4000, which is equivalently written 4×10^3, where 10^3 is the *order of magnitude*. For another example, seven microseconds is equal to $0.000007\,sec$, or equivalently, $7.0 \times 10^{-6}\,sec$. Note in each case the whole notion of the 'size' of each number is contained in the order of magnitude.

The main advantage to using power-of-ten notation for magnitudes is we don't have to waste time writing zeros. Many of the powers of ten have a special name. These are listed below in order from small to large, with the most common entries set in bold.

Power	**Prefix**	**Abbreviation**
10^{-24}	yocto	y
10^{-21}	zepto	z
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	**nano**	**n**
10^{-6}	**micro**	μ **(Greek 'myu')**
10^{-3}	**milli**	**m**
10^{-2}	**centi**	**c**
10^{-1}	**deci**	**m**
10^{3}	**kilo**	**k**
10^{6}	**mega**	**M**
10^{9}	**giga**	**G**
10^{12}	**tera**	**T**
10^{15}	peta	P
10^{18}	exa	E
10^{21}	zetta	Z
10^{24}	yotta	Y

Examples

Order	**Example**
10^{-30}	Mass of an electron in kilograms.
10^{-27}	Mass of a hydrogen atom in kilograms.
10^{-15}	Diameter of a hydrogen atom in meters.
10^{-11}	Radius of a hydrogen atom in meters.
10^{-9}	Volume of a flea in cubic meters.
10^{-6}	Lifetime of a muon particle in seconds.
10^{-3}	Oscillation period of a guitar string in seconds.
10^{0}	Height of a human child in meters.
10^{3}	Mass of a car in kilograms.
10^{6}	Radius of Earth in meters.
10^{9}	Duration of a century in seconds.
10^{12}	Number of cells in human body.
10^{16}	Radius of the solar system in meters.
10^{27}	Mass of Jupiter in kilograms.
10^{30}	Mass of the Sun in kilograms.

Significant Figures

Insignificant Figures

A number's precision is subject to the methods or apparatus used to attain that number. For instance, a length measured by a ruler might be precise to the one millimeter range, which means two things: (i) Any object smaller than one millimeter won't be 'visible' to the ruler (requiring a finer instrument). (ii) Any length measured by the ruler cannot be known to greater precision than $1\,mm = 0.1\,cm = 0.001\,m$. To illustrate, suppose your lab assistant measures his shoelace with the above-mentioned ruler and reports the result $L = 26.5912\,cm$. You should immediately round this to $L = 26.6\ cm$, because the 'junk' digits 912 occupy decimal places that are smaller than the precision of the ruler.

A way to justify throwing away the 'junk' digits is to consider what happens when the measurement is repeated. While it's highly likely that the length L will be close to $26.6\,cm$ again, it's highly unlikely that the digits 912 would follow a second time. In fact, we aren't confident that the shoelace really is $26.6\,cm$ until this value is compared to the average from multiple measurements.

Significant Figures

The precision in a number is represented by how many decimal places are *confidently* known, where each decimal place is called a *significant figure* or *significant digit*. Supposing a quantity is handed to us *without* any junk digits, there are several rules for counting the number of significant figures. These are:

1. Zeros to the left of the number are not significant. (For example, \$00237.66 is the same as \$237.66.)

2. Zeros to the right of the number are significant. (More zeros mean more precision.)

3. All nonzero digits in the number are significant.

4. Any zeros between significant digits are significant.

The Caveat

Sometimes the precision of a number is not obvious. If someone says they earned \$4500 last month, does this precisely mean \$4500.00 dollars and zero cents, or was \$4500 rounded from some nearby number? Should we trust only the 4 and the 5?

This issue is resolved by the following convention: *When the precision is ambiguous, do intermediate calculations to four significant figures, and report answers to three significant figures.* For the example on hand, we can agree that an honest representation of \$4500 is $\$4.50 \times 10^3$.

Propagation of Significant Figures

When two numbers of finite precision are combined, the natural question arises: what's the precision of the result? The answer depends on the operation being performed, but there are two main ways to proceed.

Multiplication, Division, Roots, Powers

Be pessimistic: In your calculation, seek the quantity with the lowest precision, and adopt *that* precision for the final answer. (This makes sense in the same way that a chain is only as strong as its weakest link.) The calculated quantity should be only as precise as the factors contributing to it. This point is illustrated in the following examples:

$$2.345 \times 5.4 = 13$$
$$55555 \times 0.0100 = 556$$
$$\sqrt{\pi \times 2.56 \div 5.567} = 1.20$$
$$70 \times 23.846251 = 1.7 \times 10^3$$
$$\left[(2/3) \times \left(6.984 \times 10^{-8}\right)\right]^2 = 2.170 \times 10^{-15}$$

Notice that purely mathematical factors like 2 or $\pi = 3.14159265358979323846264\ldots$ or $2/3 = 0.666\bar{6}$ are more-or-less infinitely precise numbers, causing no penalty in the overall precision of a calculation. Calculators assume all numbers are pure so watch out.

Addition and Subtraction

If the quantities being added or subtracted do not involve any decimals, you're off the hook and may combine as normal. When there *are* decimals though, be pessimistic again: the number of decimal places carried by the answer is determined by the contributing factor with the *least* number of decimal places. See the examples that follow.

$$12551 + 3114 = 15665$$
$$1000 - 3 = 997$$
$$2.345 + 5.4 = 7.7$$
$$100.2 + 15.438 = 115.6$$
$$1000 - 0.3 = 1000$$

Scientific Notation

Scientific notation is a standard for expressing numbers with the power of 10 separated from the base number, which is a handy because both the size and the precision of a number are each plainly obvious. See the examples in the table below.

Quantity	**Sig. Figs.**	**Sci. Not.**
7	1	7×10^0
10^{-3}	1	1×10^{-3}
3.0×10^8	2	3.0×10^8
000.0000010	2	1.0×10^{-6}
2.998×10^8	4	2.998×10^8
0623.53	5	6.2353×10^2
12.340	5	1.2340×10^1
90000	5	9.0000×10^4
90000.00	7	9.000000×10^4

3 Units in Quantum Mechanics

The fundamental constants in quantum theory, expressed in terms of Planck's length $[L]$, mass $[M]$, and time $[T]$ are:

- the speed of light c: $[LT^{-1}]$
- Planck's reduced constant $\hbar = h/2\pi$: $[ML^2T^{-1}]$
- the squared electron charge $e^2/(4\pi\epsilon_0)$: $[ML^3T^{-2}]$
- the electron mass m: $[M]$

Reduced Compton Wavelength

As a useful exercise, we may determine x, y, and z such that

$$\hbar^x c^y m^z = [L] \ ,$$

i.e., has dimensions of length. Using dimensional analysis, are handed three equations

$$x + z = 0 \qquad 2x + y = 1 \qquad x + y = 0 \,,$$

solved by $x = 1$ and $y = z = -1$, telling us that

$$\lambda_c = \hbar/mc \,,$$

known as the *reduced Compton wavelength* λ_c, which evaluates to roughly $0.386 \times 10^{-12}\ m$.

Fine Structure Constant

The combination

$$\hbar^x c^y (e^2/4\pi\epsilon_0)^z$$

can yield a dimensionless quantity. In this case, we generate three equations

$$x + z = 0 \qquad 2x - 1 + 3z = 0 \qquad -x + 1 - 2z = 0 \,,$$

solved by $x = -1$, $z = 1$. Choosing $y = -1$ with the remaining freedom, we get the *fine structure constant*:

$$\alpha = \frac{e^2/4\pi\epsilon_0}{\hbar c} \approx \frac{1}{137} \approx 0.00730$$

Bohr Radius

We can recover the *Bohr radius* by taking the ratio of the reduced Compton wavelength to the fine structure constant:

$$a_0 = \frac{\lambda_c}{\alpha} = \frac{\hbar^2}{m\,(e^2/4\pi\epsilon_0)} \approx 137 \times \lambda_c \approx 5.29 \times 10^{-11}\ m$$

4 Large Numbers

Following our section on order of magnitude, it's clear why big numbers like 10^{30} are called 'astronomical'. However, these still pale in comparison to 'thermodynamically' large numbers, such as 999^{999}, or the factorial of 1000. To understand the size of such numbers, the primary task is to convert to scientific notation.

Large Exponents

Proceeding for large exponents, suppose we are handed the exponent X^Y, where Y is not limited to 10. It must follow that

$$X^Y = A \times 10^B \ ,$$

where A is the leading coefficient limited to $0 \leq A < 10$, and B is the order of magnitude, defined as an integer. The coefficient A can be written as 10^C, where $0 \leq C < 1$, giving

$$X^Y = 10^C \times 10^B = 10^{C+B} \ .$$

Take the natural log of both sides and separate knowns from unknowns.

$$\begin{aligned} \ln\left(X^Y\right) &= \ln\left(10^{C+B}\right) \\ Y \ln X &= (C+B) \ln 10 \\ \frac{Y \ln X}{\ln 10} &= C + B \end{aligned}$$

Of course, it appears that there are still two unknowns C and B, but remember these are already constrained: B is an integer, and C is less than one. Defining Q such that

$$Q = \frac{Y \ln X}{\ln 10} \ ,$$

along with an 'integer' function $\text{int}(z)$ that always rounds down to nearest integer, it readily follows that

$$Q = (Q - \text{int}(Q)) + \text{int}(Q) \ ,$$

which simultaneously tell us C and B. We must have

$$C = Q - \text{int}(Q) \qquad\qquad B = \text{int}(Q) \ .$$

Finally, we stitch the answer together:

$$X^Y = 10^{Q - \text{int}(Q)} \times 10^{\text{int}(Q)}$$

Example

To evaluate 999^{999}, we have

$$Q = \frac{999 \ln 999}{\ln 10} = 2996.5659227377 \ ,$$

implying

$$C = 0.5659227377 \qquad\qquad B = 2996 \ .$$

Evaluating $A = 10^C$, we find $A = 3.6806348826$, so the result is

$$999^{999} = 3.6806348826 \times 10^{2996} \ .$$

Large Factorials

The number $N!$ for integer N is defined as N multiplied by every integer less than N, all the way down to one:

$$N! = N(N-1)(N-2)\cdots(N-(N-1)) = \prod_{j=0}^{N-1}(N-j) \ ,$$

which can get out of hand quite quickly for increasing N. To proceed, take the natural log of both sides, transforming the product to a sum:

$$\ln(N!) = \sum_{j=0}^{N-1}\ln(N-j) \ ,$$

and then undo the log by taking exp() of both sides:

$$N! = e^{\ln N!} = \exp\left(\sum_{j=0}^{N-1}\ln(N-j)\right) = e^{R} \ ,$$

which converts the factorial into the exponential of a sum, contained in the variable R, where

$$R = \sum_{j=0}^{N-1}\ln(N-j) \ .$$

The task is reduced to converting the term e^R into the format $A \times 10^B = 10^{C+B}$, the precise case solved above. Borrowing the result, we write

$$C + B = Q = \frac{R\ln e}{\ln 10} = \frac{R}{\ln 10} \ ,$$

where

$$C = Q - \text{int}(Q) \qquad\qquad B = \text{int}(Q) \ ,$$

and finally

$$N! = e^R = 10^{Q-\text{int}(Q)} \times 10^{\text{int}(Q)} \ .$$

Example

To evaluate 1000!, we first need to evaluate R (preferably not by hand), which comes to

$$R = \sum_{j=0}^{1000-1}\ln(1000-j) = 5912.128178488164 \ ,$$

where dividing by $\ln 10$ gives Q, namely

$$Q = \frac{R}{\ln 10} = 2567.604644222133 \ ,$$

implying

$$C = 0.6046442221 \qquad\qquad B = 2567 \ .$$

Evaluating $A = 10^C$, we find $A = 4.0238726008$, so the result is

$$1000! = 4.0238726008 \times 10^{2567} \ .$$

Stirling's Approximation

Factorials of large numbers obey an approximation that eliminates the factorial (!) symbol called *Stirling's approximation*:

$$\ln(n!) \approx n\ln(n) - n + \ln\left(\sqrt{2\pi n}\right) \qquad\qquad n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

To derive Stirling's approximation, begin with the gamma function identity

$$\Gamma(z) = (z-1)! = \int_0^\infty t^{z-1} e^{-t}\, dt$$

to write

$$n! = \int_0^\infty e^{-x} x^n \, dx \, .$$

Using plots of the sharply-peaked function $e^{-x}x^n$, or by calculating its derivative, one may easily argue that the approximation $x \approx n + \epsilon$ holds for $n \gg \epsilon$. Following this, we find

$$e^{-x}x^n \approx e^{-(n+\epsilon)}(n+\epsilon)^n$$
$$\ln\left(e^{-x}x^n\right) \approx -n - \epsilon + n\ln(n+\epsilon) \ ,$$

where the expansion for natural logarithms

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \dots \qquad\qquad y \ll 1$$

leads us to

$$n! \approx \int_0^\infty \exp\left(n\ln n - n - \frac{\epsilon^2}{2n} + \cancel{\frac{\epsilon^3}{3n^2}} - \dots\right) d\epsilon \ ,$$

noting that orders of ϵ greater than two are neglected. The above resolves to

$$n! \approx e^{n\ln n - n} \int_0^\infty e^{-\epsilon^2/2n}\, d\epsilon \ ,$$

and since the integrand is essentially zero away from the peak, we extend the integration limit from 0 to $-\infty$. This transforms the final calculation to a Gaussian integral, obeying

$$\int_{-\infty}^\infty e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}} \ ,$$

which reproduces the top equation(s),

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \, .$$

5 Problems

Problem 1

The speed limit on a Canadian highway is 120 kilometers per hour. Convert this speed to miles per hour.

Solution 1

$$120\,\frac{km}{hr} = 120\frac{\cancel{km}}{hr}\left(\frac{1\,mi}{1.609\,\cancel{km}}\right) = \frac{120}{1.609}\frac{mi}{hr} = 74.6\,mph$$

Problem 2

The speed of light in vacuum is $3.00\ \times 10^8\ m/s$. Convert this speed to miles per second and also miles per hour.

Solution 2

$$\begin{aligned} 3.00\times 10^8\,\frac{m}{s} &= 3.00\times 10^8\,\frac{\cancel{m}}{s}\left(\frac{100\,\cancel{cm}}{\cancel{m}}\right)\left(\frac{1\,\cancel{in}}{2.54\,\cancel{cm}}\right)\left(\frac{1\,\cancel{ft}}{12\,\cancel{in}}\right)\left(\frac{1\,mi}{5280\,\cancel{ft}}\right) \\ &= 1.86\times 10^5\,\frac{mi}{s} \\ &= 1.86\times 10^5\,\frac{mi}{\cancel{sec}}\left(\frac{3600\,\cancel{sec}}{1\,hour}\right) = 6.71\times 10^8\,mph \end{aligned}$$

Problem 3

Estimate your age in seconds by starting with years (rounded to the nearest tenth) and multiplying that number by factors of 1 until you have the answer.

Solution 3

Assuming your age is twenty years exactly:

$$Age = 20.0\,yr = 20.0\,\cancel{yr}\left(\frac{365.25\,\cancel{day}}{\cancel{yr}}\right)\left(\frac{24\,\cancel{hr}}{1\,\cancel{day}}\right)\left(\frac{3600\,sec}{1\,\cancel{hr}}\right) = 6.31\times 10^8\,s$$

Problem 4

The thickness of paper is closest to (choose one):

$10^{-4}\,m$ $\qquad 10^{-1}\,m$ $\qquad 10^{1}\,cm$ $\qquad 10^{-7}\,m$

Solution 4

Paper thickness is closest to one tenth of a millimeter, or $10^{-4}\,m$. The rest are way off.

Problem 5

The length of a football field (100 yards) is closest to (choose two):

$10^{4}\,m$ $\qquad 10^{-1}\,km$ $\qquad 10^{5}\,mm$ $\qquad 10^{3}\,foot$

Solution 5

The two reasonable answers (are actually equivalent):

$$10^{-1}\,km = 10^6\,mm$$

<u>Problem 6</u>

Three gallons of paint are uniformly spread over a wall having area 600 ft^2. Calculate the thickness of the paint in millimeters.

<u>Solution 6</u>

Need two ways to express the volume of paint. We have $V = 3.00\,gal$, or equivalently, $V =$ thickness $\times$ area. Eliminating V and solving for thickness T, find:

$$\begin{aligned} T &= \frac{3.00\,gal}{600\,ft^2} = \frac{3.00\,gal}{600\,ft^2}\left(\frac{3.785\,L}{1\,gal}\right)\left(\frac{1000\,mL}{1\,L}\right)\left(\frac{1\,cm^3}{1\,mL}\right) = 18.925\left(\frac{cm^3}{ft^2}\right) \\ &= 18.925\left(\frac{cm^3}{ft^2}\right)\left(\frac{1\,ft}{12\,in}\right)^2\left(\frac{1\,in}{2.54\,cm}\right)^2\left(\frac{10\,mm}{1\,cm}\right) = 0.204\,mm \end{aligned}$$

<u>Problem 7</u>

The radius of Earth has order of magnitude 10^6 meters. If the volume of a sphere is given by $V = (4/3)\pi R^3$, roughly estimate the order of magnitude of Earth's volume in cubic meters.

<u>Solution 7</u>

Earth is actually a bit bigger than this, but the rough estimation is still valid: $V \approx (4/3)\pi(10^6\,m)^3 \approx 4\,(10^6\,m)^3 \approx 10^{18}\,m^3$.

<u>Problem 8</u>

The mass of Earth is approximately $M = 5.972 \times 10^{24}\,kg$. Using radius $R = 6371\,km$, calculate the average density (total mass over total volume) of Earth in grams per cubic centimeter.

<u>Solution 8</u>

Density is mass over volume:

$$\begin{aligned} \rho &= \frac{M}{V} = \frac{M}{(4/3)\,\pi R^3} = \frac{5.972 \times 10^{24}\,kg}{(4/3)\,\pi\,(6371\,km)^3} = 5.513 \times 10^{12}\,\frac{kg}{km^3} \\ &= 5.513 \times 10^{12}\frac{kg}{km^3}\left(\frac{1\,km}{1000\,m}\right)^3\left(\frac{1\,m}{100\,cm}\right)^3\left(\frac{1000\,g}{kg}\right) = 5.513\,\frac{g}{cm^3} \end{aligned}$$

<u>Problem 9</u>

At sea level, a 2.205 pound rock weighs the same as a 1 kilogram object. On Mt. Everest, a 2.199 pound rock weighs the same as a 1 kilogram object. What is the weight in pounds of an 80 kg person at each location?

<u>Solution 9</u>

At sea level, 80 $kg \sim 176.4\,lb$. On Mt. Everest, 80 $kg \sim 175.9\,lb$. There's about a half-pound difference.

Problem 10

In standard conditions, water freezes at $0^\circ\ C = 32^\circ\ F$, and it boils at $100^\circ\ C = 212^\circ\ F$. The two temperature scales (Fahrenheit and Centigrade) obey a linear relation

$$T_F = m \cdot T_C + b\ .$$

Determine m and b (as in $y = mx + b$).

Solution 10

Generate two equations and two unknowns:

$$32^\circ F = m \cdot 0 + b \qquad\qquad 212^\circ F = m \cdot 100^\circ C + b$$

First eliminate b to solve for m:

$$212^\circ F = m \cdot 100^\circ C + 32^\circ F \qquad \rightarrow \qquad m = \frac{180^\circ F}{100^\circ C}$$

Next solve for b, and write the final answer:

$$b = 32^\circ F \qquad \rightarrow \qquad T_F = \frac{9}{5}\,[T_C] + 32^\circ F$$