Conic Sections

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1 Introduction

A *conic section* is the set of points defined by the intersection of a cone and a plane. The manner in which the intersection occurs determines the species of conic section produced. A perpendicular cross section of a cone results in a **circle** intersection, where if we tilt the plane slightly, the circle stretches to an **ellipse**. Continue tilting the plane until parallel with the cone to produce a **parabola**, and continue tilting further to produce a **hyperbola**.

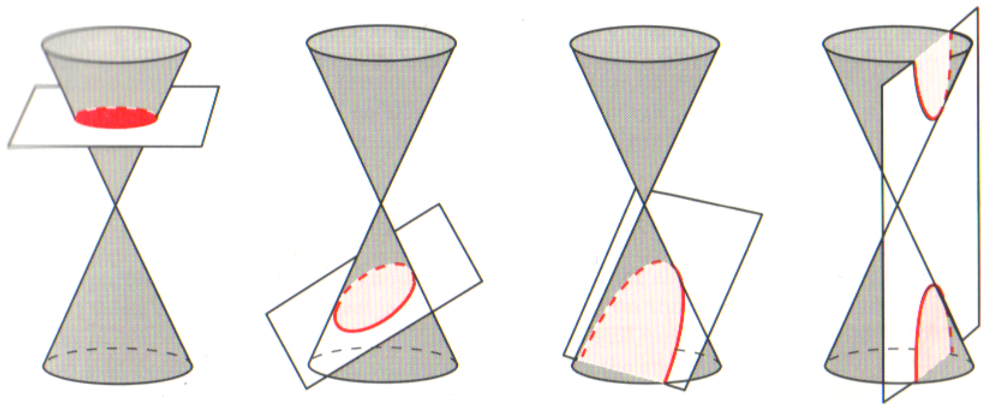

Figure 1: Four conic sections. (Image credit: *Calculus*, Gilbert Strang)

Conic sections, or *conics* for short, are ubiquitously common in physics and engineering. Conics are also invaluable as a pedagogical tool for connecting algebra to geometry, which shall be the slant of this document.

The Four Conics

In the xy-plane, the equation of a circle centered at the origin is given by

$$x^2 + y^2 = r^2 \, ,$$

where the radius r is a constant. If we stretch out the circle to have 'width' and 'height' parameters a and b, the governing equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \, .$$

For the special case $a = b$, the equation of the ellipse reduces to that of a circle.

The parabola is a special conic section in that one of the x- or y-variables occurs to the first power, whereas the other always occurs as a square. An up- or down-opening parabola has the form

$$y = ax^2 \, ,$$

whereas a left- or right-opening parabola has the form

$$x = ay^2 \, .$$

Hyperbolic curves are rich with detail, all of which arises from reversing one of the signs in the equation of the ellipse. Indeed,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad\qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represent left-right-opening and up-down-opening hyperbolic curves, respectively.

2 Slicing the Cone

Diving right in, let us consider a cone defined by the equation

$$z^2 = x^2 + y^2 \; .$$

Circle

Such a cone can be viewed as a continuous stack of concentric circles, where a given circle's height from the xy-plane is equal to its radius. This means we derive the 'circle' case for free, which of course encloses the trivial point case $x = y = z = 0$.

To cover the non-circle cases, we need an adjustable plane to slice the cone, with a general-enough choice being

$$z = 1 - \frac{x}{c}$$

as shown, with the parameter c determining the angle θ between the plane and the x-axis.

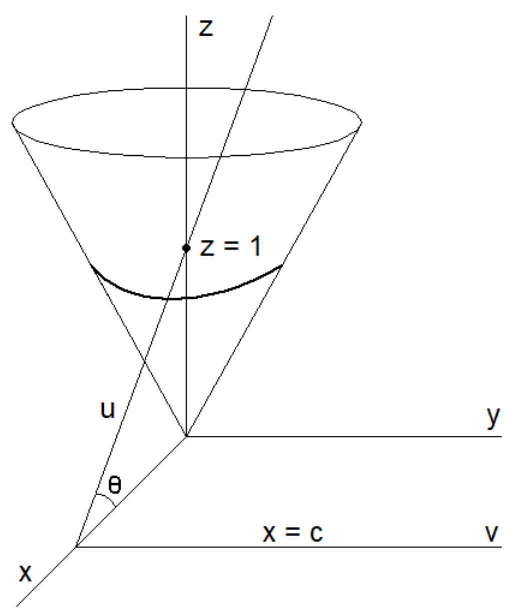

Figure 2: Slicing the cone.

The plane $z = 1 - x/c$ carries embedded x-like and y-like axes, which we label u and v, respectively, with the trivial observation that $v = y$. With this construction, the intersection between the cone and the plane - literally the conic section - will occur as some shape in the uv-plane.

To proceed, we first relate all variables to u, v, and θ, giving

$$u\cos\theta = c - x \qquad u\sin\theta = z \qquad \rightarrow \qquad \tan\theta = \frac{1}{c}\ ,$$

leaving a concise formula for intersections

$$z = 1 - \tan\theta\sqrt{z^2 - y^2}\ .$$

Parabola

The special case $\theta = \pi/4$ corresponds to the slant of the plane equaling that of the cone, reducing the above to

$$\cancel{z^2} - 2z + 1 = \cancel{z^2} - y^2\ ,$$

reducing further to an unmistakable equation for a parabola:

$$u = \frac{v^2 + 1}{\sqrt{2}}$$

Ellipse

Replacing all instances of z and y in favor of the u- and v-variables, the intersection formula becomes

$$u\sin\theta = 1 - \tan\theta\sqrt{u^2\sin^2\theta - v^2}\ ,$$

simplifying further to

$$\left(u\cos\theta\left(1 - \tan^2\theta\right) - \cot\theta\right)^2 + v^2\left(1 - \tan^2\theta\right) = 1\ .$$

Denoting $\gamma = 1 - \tan^2\theta$, we have

$$\left(u\cos\theta \cdot \gamma - \cot\theta\right)^2 + v^2 \cdot \gamma = 1\ ,$$

which represents an ellipse in the vu-plane for positive γ (that is, $\theta < \pi/4$).

Hyperbola

If we allow θ to increase beyond $\pi/4$, the sign on γ reverses. This ultimately causes the u^2-term and the v^2-term to differ in sign, giving rise to hyperbolic conic sections.

3 Directrix and Focus

The four conic sections can be recovered from an entirely two-dimensional pursuit using a few out-of-the-hat rules. To begin, consider a vertical line called in the xy-plane called the *directrix*. Place an arbitrary point on the x-axis as shown called the *focus*.

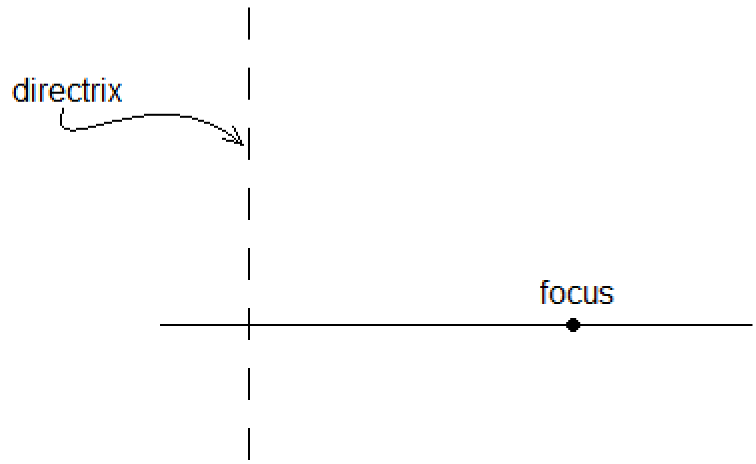

Figure 3: The directrix and the focus.

Parabola

A parabola can be defined as the family of points in the plane such that *each point's distance to the focus is equal to the (shortest) distance to the directrix.*

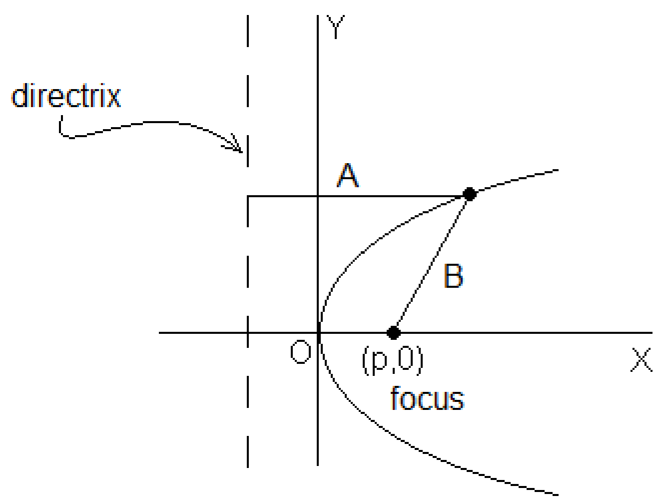

Figure 4: Constructing the parabola from directrix/focus.

Let the directrix lay at $x = -p$, and the focus at $x = p$. To capture the definition of the parabola, we write

$$B = A \qquad\qquad \to \qquad\qquad \sqrt{(x-p)^2 + y^2} = p + x \,,$$

which easily reduces to an equation of a parabola,

$$y^2 = 4px \,,$$

where $2p$ is the shortest distance between the directrix and the focus.

Ellipse (Cartesian Coordinates)

A similar definition involving a directrix and a focus produces an ellipse. This time rather, we seek *the set of points such that the distance to the focus divided by the distance to the directrix is a constant* $e < 1$ (not simply unity as for the parabola).

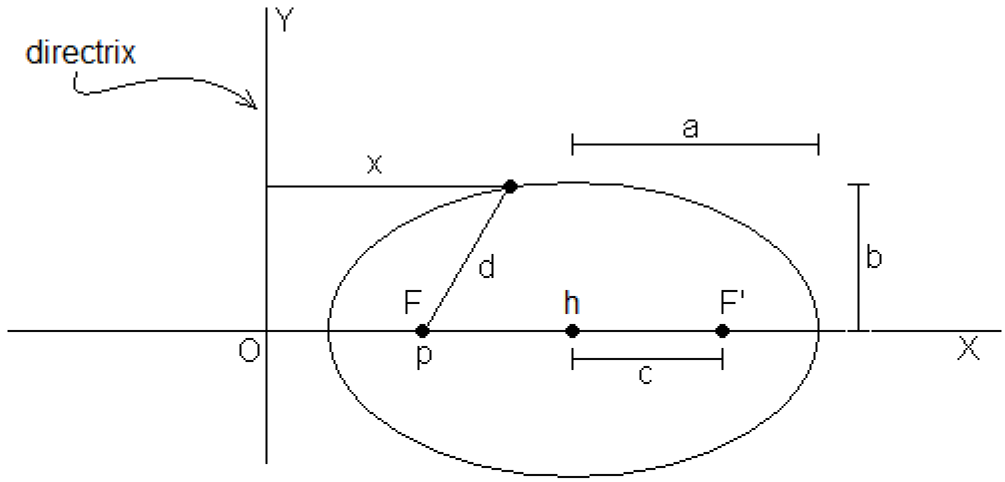

Figure 5: Constructing the ellipse from directrix/focus in Cartesian coordinates.

Anticipating the elliptical result, the semi-major axis a, semi-minor axis b, and center h of the ellipse are depicted. The focus F, located at p, has a symmetric partner F', each distance c from the center. The directrix lies along the Cartesian y-axis.

Following the proposed definition, we begin with

$$\frac{d}{x} = e \qquad \rightarrow \qquad \frac{\sqrt{(x-p)^2 + y^2}}{x} = e \, ,$$

which we aim to wrestle into an ellipse-like equation. This is most easily done by squaring both sides and completing the square in x to eventually arrive at

$$\frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1 \, ,$$

where

$$h = \frac{p}{1-e^2} \qquad a = \frac{ep}{1-e^2} \qquad b = \frac{ep}{\sqrt{1-e^2}} \, .$$

It's useful to eliminate p between the a- and b- equations, giving

$$\frac{b}{a} = \sqrt{1-e^2} \qquad e = \sqrt{1 - \frac{b^2}{a^2}} \, .$$

The 'internal' length c is directly proportional to e, as

$$h = p + c \qquad \rightarrow \qquad c = ea \, ,$$

which by eliminating e, reduces to

$$c = \sqrt{a^2 - b^2} \, .$$

Ellipse (Polar Coordinates)

The above results can be naturally attained in polar coordinates by placing the origin of the system af the focus F.

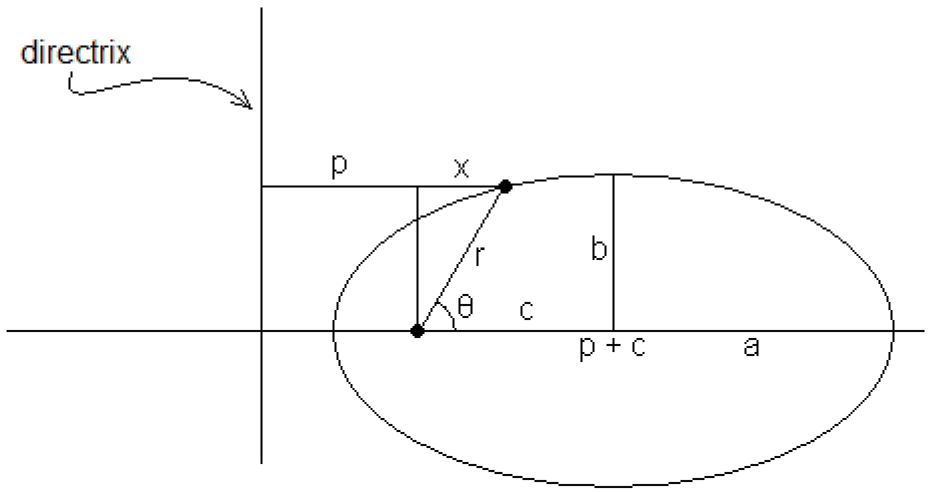

Figure 6: Constructing the ellipse from directrix/focus in polar coordinates.

With distance d replaced by the radius $r(\theta)$, the definition of the ellipse reads

$$\frac{r}{x} = e \qquad\qquad x - p = r\cos\theta \,,$$

allowing x to be eliminated to give the equation of an ellipse:

$$r(\theta) = \frac{ep}{1 - e\cos\theta}$$

Of course, the polar form is equivalent to the Cartesian form, which is shown by noting $y = r\sin\theta$, allowing θ to be eliminated:

$$\cos^2\theta + \sin^2\theta = \left(\frac{x-p}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \,.$$

After replacing r and p, the familiar h-shifted form emerges.

Hyperbola

Finally, we tweak our directrix-and-focus construction to produce a hyperbola by seeking *the set of points such that the distance to the focus divided by the distance to the directrix is a constant* $e > 1$.

The following calculation proceeds much as the elliptical case, up to important differences caused by $e > 1$, namely

$$a \to -a \qquad\qquad b \to \sqrt{-1}\,b = ib \qquad\qquad h \to -h = k \,.$$

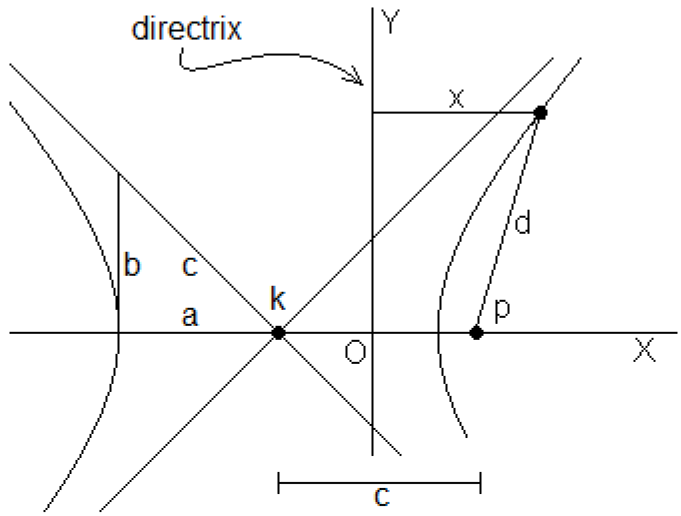

Figure 7: Constructing the hyperbola from directrix/focus in Cartesian coordinates.

Right away, the hyperbola must be described by

$$\frac{(x+k)^2}{a^2} - \frac{y^2}{b^2} = 1 \; .$$

Looking closely, the hyperbola consists of two separate curves, which is equivalent to acknowledging that both the cone and its conjugate are sliced by the same plane. Like the ellipse, the hyperbola also has two focus points and two directrix lines.

Two straight lines with slopes $\pm b/a$ are placed on the graph to illustrate an important feature: the hyperbola is *asymptotic* to these lines. That is, the hyperbola 'opens up' at a rate that approaches a constant as $x \to \infty$. This is very much unlike the parabola, whose slope grows forever with x.

The hyperbola's center-to-vertex distance is a, and the center-to-focus distance is $c = ea$. Length b is the line that extends tangent to the vertex and hits the asymptotic line. Point p (the focus) is located at $p = a\left(e^2 - 1\right)/e$. The internal lengths of the hyperbola obey the relation $a^2 + b^2 = c^2$, and the axes relate to the eccentricity by $b^2/a^2 = e^2 - 1$.

4 Geometric Properties

Review

Condensed in the following table are the essential aspects of each conic section.

	equation	vertex	focus
parabola	$y = ax^2 + bx + c$	$(-b/2a, c - b^2/4a)$	$1/4a$ above the vertex
ellipse	$y^2/a^2 + x^2/b^2 = 1$	$(a, 0)$, $(-a, 0)$	$(c, 0)$, $(-c, 0)$; $c^2 = a^2 - b^2$
hyperbola	$y^2/a^2 - x^2/b^2 = 1$	$(0, a)$, $(0, -a)$	$(0, c)$, $(0, -c)$; $c^2 = a^2 + b^2$

Conic sections are (perhaps) more transparent when expressed in polar coordinates *with the origin at a focus*, having form

$$r(\theta) = \frac{ep}{1 - e\cos\theta} \,.$$

This is familiar from our study of the ellipse, however is general enough to represent any conic section. To place the origin at the other (right-hand) focus, reverse the sign on the cos-term.

Eccentricity

The constant e is called the *eccentricity*. The special case $e = 0$ corresponds to a circle, where r reduces to a constant. For another special case $e = 1$, the corresponding conic is parabolic. All curves with $0 < e < 1$ correspond to ellipses, whereas $e > 1$ corresponds to the hyperbolic case.

Generalized Circle

An ellipse may be regarded 'generalized circle'. Imagine you hold a thin closed loop of string (total length L), a pencil, and a pin firmly attached to a flat surface. To trace a circle, place the loop of string onto the surface so that it captures pin, and then use the pencil point to pull the loop tight. Maintaining constant tension in the string, move the pencil completely around the pin while marking the surface. The resultant shape is a circle of diameter L. If you like, interpret the center of the circle as its single focus.

Let us now modify the construction by fixing *two* pins (separated by $2c$) onto the flat surface, which we take as the two focus points. The tracing is the same: place the loop around the two pins, and use the pencil point to maintain a constant tension in the string while marking the surface. The emerging shape is suspiciously elliptical.

Ellipse

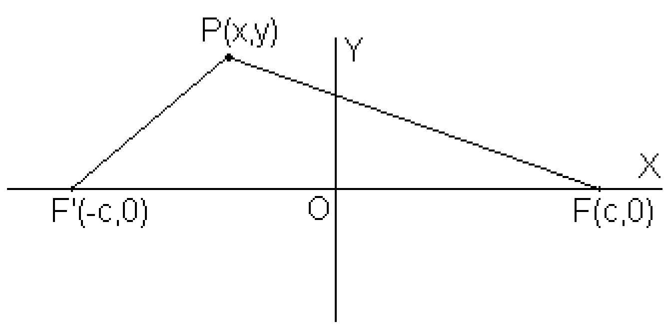

Figure 8: Generation of the ellipse using focus points and a tight string.

We know by construction that $F'P + PF + 2c = L$ as depicted in the Figure. The special case $x = \pm a$, $y = 0$ further indicates $2a + 2c = L$, which combine to eliminate L as

$$F'P + PF = 2a \,.$$

In terms of Cartesian coordinates, the above reads

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a\ ,$$

readily reducing to the equation of a centered ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ,$$

which requires all instances of a^2-c^2 to be replaced by b^2.

Evidently, we have discovered a definition of the ellipse that reads: *the set of points in a plane such that the sum of the distances to two fixed points is constant.*

Hyperbola

Following the alternative definition of the ellipse, let us propose that *the set of points in a plane such that the difference of the distances to two fixed points is constant.* Such a claim is contained in

$$\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}=2a\ ,$$

readily reducing to the equation of a centered hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ ,$$

which requires all instances of c^2-a^2 to be replaced by b^2.

5 Reflection Properties

Parabolic Mirrors

Conic sections are enormously common in physics and engineering applications. For example, suppose a design is needed for a curved mirror that reflects rays of light from an isotropic point source into one and the same direction (like a car headlight). Or equivalently, the same mirror would be needed to converge incoming parallel rays onto a single point (like a satellite receiver). It turns out that a parabola solves both of these problems.

In the above Figure, the origin is at the light source, and we draw a ray emerging off-axis, reflecting off of a wall (our soon-to-be parabolic mirror), and projecting to $x\to\infty$. From the picture, observe

$$\tan 2\theta=\frac{y}{x}\qquad\qquad \frac{dy}{dx}=\tan\theta\ ,$$

and recall from trigonometry that

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}\ .$$

Eliminating all θ-terms in favor of variables x and y, arrive at the differential equation

$$y\,(y')^2+2xy'-y=0\ ,$$

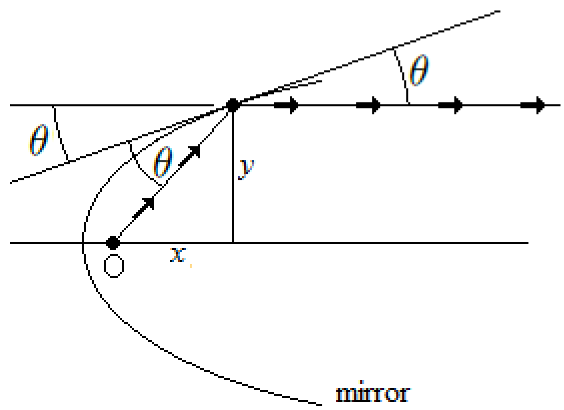

Figure 9: The parabola as a reflector.

where y' is the slope dy/dx. To solve this, make the change of variables $y^2 = r^2 - x^2$ (so $yy' = rr' - x$). The problem reduces to solving $r' = 1$, which has solution $r = x \pm c$. Substitution back into $y^2 = r^2 - x^2$ gives the right-facing parabola we wanted:

$$y^2 = c\,(c \pm 2x)$$

Elliptical Offices

Ellipses exhibit the remarkable property that *any ray emerging from a focus intercepts the other focus after one reflection*. As a restatement of the geometric definition of the ellipse, it follows that all reflected rays take the same time to reach the other focus.

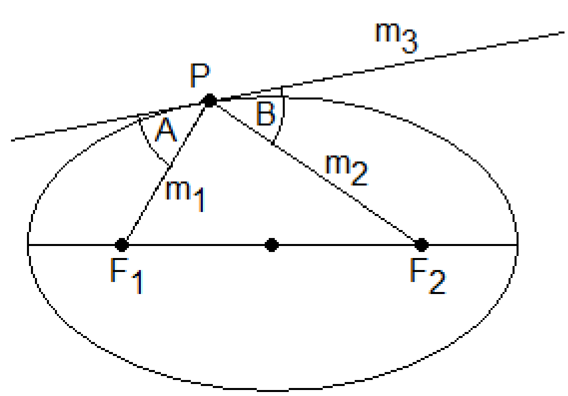

Figure 10: Ellipse with inner-reflected ray.

In the Figure we track the path of a single ray that emerges from focus F_1 with slope m_1 that then intersects the ellipse at point $P = (X, Y)$, making angle A with the tangent line (slope m_3) at P. The reflected ray (slope m_2) heads toward F_2 making angle B with the tangent line.

Proving the reflection property reduces to showing that angle A equals angle B. Observe form trigonometry that

$$\tan A = \frac{m_1 - m_3}{1 + m_1 m_3} \qquad \tan B = \frac{m_2 - m_3}{1 + m_2 m_3},$$

where the slopes m_i are

$$m_1 = \frac{Y}{X + c} \qquad m_2 = \frac{Y}{X - c} \qquad m_3 = -\frac{b^2}{a^2}\frac{X}{Y}.$$

The formula for m_3 comes from implicit differentiation of $(x/a)^2 + (y/b)^2 = 1$. The rest of the proof is now a matter of algebra. Simplifying the equations for $\tan A$ and $\tan B$ give, respectively,

$$|\tan A| = \frac{b^2}{cY} \qquad |\tan B| = \frac{b^2}{cY},$$

indicating $A = B$, finishing the job.

Trapping Rays

The hyperbola harbors its own geometric coincidences that echo with those of the ellipse. In analogy to the reflection property, it turns out that *any incoming ray aimed at a focus will be reflected to the other focus.*

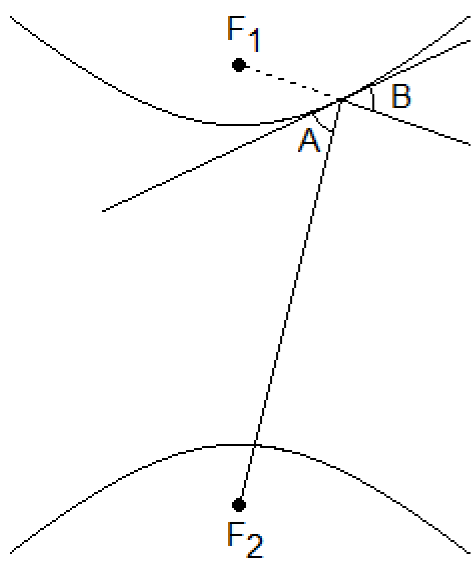

Figure 11: Hyperbolic mirror. A ray from the right is directed to F_2 after one reflection.

In the above Figure, a ray aimed at F_1 reflects from a hyperbolic mirror and advances to F_2. Proving this amounts to showing that angles A and B are equivalent, a near-identical procedure to the elliptical case.

Locating Ships at Sea

As a corollary to the geometric definition of the hyperbola, we notice that *a signal emitted from any point on a hyperbola reaches the respective focus points a fixed time apart.* The time interval is given by $2a/v$, where v is the propagation speed.

To make use of this, consider two radio stations separated by distance $d = 2c$ on land. A signal emitted from a nearby ship reaches each station across the interval $\Delta t = 2a/v$. Using the internal relation $a^2 + b^2 = c^2$, we solve for b to get

$$b = \pm\frac{1}{2}\sqrt{d^2 - v^2\Delta t^2}\ ,$$

which confines the ship's location to *somewhere* on a known hyperbola. If a third radio station is involved, ship's exact location is the overlap of two hyperbolas.

6 Generalized Conic Sections

In real life, conic sections may not appear as simple formulas like $y^2 = 4px$, especially if the axes are tilted or rotated. To illustrate the method for taking hold of these situations, start with the most general conic section,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\ .$$

Classifying Conics

Hyperbolas have $B^2 > 4AC$ and ellipses have $B^2 < 4AC$. If it just happens that $B^2 = 4AC$, then the curve is a parabola. This can be seen by solving for y using the quadratic formula - a step that doesn't indicate the orientation of the conic, but at least the shape is classified.

Rotations

To proceed in the general case, complete the square to eliminate the linear terms, giving

$$\left(\sqrt{A}x + \frac{D}{2\sqrt{A}}\right)^2 + Bxy + \left(\sqrt{C}y + \frac{E}{2\sqrt{C}}\right)^2 - \frac{D^2}{4A} - \frac{E^2}{4C} + F = 0.$$

At this point, it's smart to shift and rescale the variables x and y, and also divide by an overall constant so the general conic takes the form

$$ax^2 + bxy + cy^2 = 1,$$

where a, b, and c are not the same as their uppercase predecessors. As before, the curve is a parabola if $b^2 = 4ac$, hyperbolas have $b^2 > 4ac$, and ellipses have $b^2 < 4ac$.

The task that remains is to squelch the xy term, which is equivalent to rotating the entire curve so it aligns 'nicely' with the coordinate system (no slant). The new curve should be expressed in new variables x' and y' with no $x'y'$ term. The tool for the job is the rotation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The inverse relation, used for rotating the coordinate system rather than the curve itself, is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Substitution of x and y into $ax^2 + bxy + cy^2 = 1$ gives the modified version,

$$a'x'^2 + b'x'y' + c'y'^2 = 1.$$

The task of eliminating the xy term has transformed into eliminating the $x'y'$ term, which is accomplished by setting $b' = 0$. Leaving the algebra to the reader, this amounts to the condition

$$\tan 2\theta = \frac{b}{a-c}.$$

In other words, you must rotate by angle θ to get a conic section to 'square off' with its coordinate system. Depending on the relative signs of a and c, the result $a'x'^2 \pm c'y'^2 = 1$ is recognizable (hopefully by now) as an ellipse for the $+$ case and a hyperbola for the $-$ case.

Points and Lines

The quickest way to suppress a conic section is to strip its constant term, setting $F = 0$ in the general equation. The graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ is actually just a single point, which amounts to slicing the cone precisely at its vertex.

Other extreme slices of the cone give lines. For instance, a vertical slice that passes through the cone's vertex forms a very tight hyperbola so that only its asymptotes ($y = \pm bx/a$) constitute the entire curve. That is, a slicing that matches the cone's slope, ordinarily giving a parabola, instead gives one line ($x^2 = 0$).

7 Problems

Parabolas

1. Sketch the parabola $x^2 - 2y - 6x = 0$, and then prove that the focus is located at $(3, -4)$ and that the directrix is located at $y = -5$.

2. The parabola $y^2 - 2ay + 2x - a^2 = 0$ has its focus on the y-axis above the origin. Find the number a and sketch the graph. Answer: $a = 1/\sqrt{2}$

3. A parabolic segment (i.e. the area bounded by a parabola and a chord perpendicular to the axis) is 32 cm high and its base is 16 cm. How far is the focus from the directrix? Answer: 1 cm

4. Prove that in a parabola the length of the chord passing through the focus making an angle θ with the axis is equal to $L/\sin^2\theta$, where L is the length of the *right focal chord*, the line that passes through the focus and is perpendicular to the axis.

5. Prove that the equation of the tangent to the parabola $y^2 = 4px$ at the point (x_1, y_1) is $yy_1 = 2p\,(x + x_1)$.

6. Prove that the area under a parabolic segment of base b and height h is $A = 2bh/3$.

7. Prove that the centroid of a parabolic segment of vertex height h is $\bar{y} = 2h/5$.

8. Find the area of the 'lens' formed between the curves

$$y = x^2 \qquad\qquad y = ax + b.$$

   Answer: $A = (1/6)\left(a^2 + 4b\right)^{3/2}$

9. Prove that the arc length of a parabolic segment of base $2a$ and height h is

$$L = \frac{a^2}{h}\int_0^{\frac{2h}{a}} dz\sqrt{1+z^2} = \sqrt{a^2+4h^2} + \frac{a^2}{2h}\sinh^{-1}\left(\frac{2h}{a}\right).$$

10. The axis of the parabola $y = x^2 - 1/4$ is tilted by 45 degrees using the focus as a pivot so that the new axis lies along the line $y = x$. Prove that the equation of the tilted parabola is

$$y = x + \frac{1}{\sqrt{2}} \pm \sqrt{2\sqrt{2}x + 1}.$$

Circles and Ellipses

1. For each ellipse

$$5x^2 + y^2 - 20x = 0 \qquad\qquad x^2 + 2y^2 + 4y - 6 = 0,$$

   find the center, endpoints of the major and minor diameters, focus points, directrix lines, and eccentricity. (Answer: $(2,0)$, $\left(2, \pm 2\sqrt{5}\right)$, $(0,0)$, $(4,0)$, $(2,\pm 4)$, $y_d = \pm 5$, $e = 2/\sqrt{5}$; $(0,-1)$, $\left(\pm 2\sqrt{2}, -1\right)$, $(0,1)$, $(0,-3)$, $(\pm 2, -1)$, $x_d = \pm 4$, $e = \sqrt{2}/2$)

2. Find the equation of the ellipse with vertices at $(3, 1)$ and $(-1, 1)$ and eccentricity $e = 2/3$. (Answer: $(x-1)^2/4 + 9(y-1)^2/20 = 1$)

3. Show that the equation of the ellipse in polar coordinates

$$r = \frac{b^2/a}{1 - e\cos\theta}$$

is equivalent to the Cartesian form

$$\frac{(x - a/e)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. For the ellipse $x^2/a^2 + y^2/b^2 = 1$, show that the tangent equation is $xx_0/a^2 + yy_0/b^2 = 1$.

5. Prove that the equation

$$\vec{r} = \hat{\imath} a\cos\theta + \hat{\jmath} b\sin\theta$$

is a parametric equation of an ellipse centered at the origin.

6. Prove that the area of the ellipse is $A = \pi ab$.

7. Prove that the centroid of a half-ellipse of base $2a$ and vertex height b is $\bar{y} = 4b/3\pi$.

8. Compute the area of (i) the largest square and (ii) the largest rectangle that just fits inside $x^2/a^2 + y^2/b^2 = 1$. (Answer: $A_i = \frac{4a^2b^2}{a^2+b^2}$, $A_{ii} = 2ab$)

9. Prove that the arc length around the ellipse is

$$L = 4a\int_0^{\pi/2} d\theta\sqrt{1 - e^2\sin^2\theta},$$

which has no closed solution in general. This is known as a *complete elliptic integral of the second kind*. For $e = 0$, show that the answer is $L = 2\pi r$, where $r = a = b$.

10. The equation

$$21x^2 + 31y^2 - \sqrt{300}xy = 144$$

describes a tilted ellipse centered at the origin (verify $b^2 < 4ac$ and sketch the graph if necessary). Determine the angle needed to un-tilt the ellipse: aligning the major axis along the x-axis and the minor axis along y. Also determine the equation of the un-tilted ellipse. (Answer: $2\theta = \tan^{-1}\left(\sqrt{3}\right)$, $\tilde{x}^2/9 + \tilde{y}^2/4 = 1$)

Hyperbolas

1. For each hyperbola

$$16y^2 - 9x^2 = 144 \qquad 12x^2 - 32y^2 - 12x + 96y + 27 = 0,$$

find the center, vertices, focus points, directrix lines, eccentricity, and asymptotes. (Answer: $(0, 0)$, $(0, \pm 3)$, $(0, \pm 5)$, $y = \pm 9/5$, $e = 5/3$; $(1/2, 3/2)$, $\left(1/2, 3/2 \pm \sqrt{3}\right)$, $\left(1/2, 3/2 \pm \sqrt{11}\right)$, $y = 3/2 \pm 3/\sqrt{11}$, $e = \sqrt{33}/3$)

2. Find the equation of the hyperbola with vertices at $(0, \pm 2)$ with asymptotes $y = \pm x/2$. (Answer: $y^2/4 - x^2/16 = 1$)

3. Find the equation of the hyperbola with focus points $(7, 0)$ and $(-1, 0)$ containing $\left(6, \sqrt{15}\right)$. (Answer: $(x-3)^2/4 - y^2/12 = 1$)

4. Find the slope of $y^2 - x^2 = 1$ at (x_0, y_0). Show that $yy_0 - xx_0 = 1$ goes through this point with the right slope (it has to be the tangent line).

5. Prove that the equation
$$\vec{r} = \hat{\imath} a \cosh\theta + \hat{\jmath} b \sinh\theta$$
is a parametric equation of a right-opening hyperbola centered at the origin.

6. The area inside a unit circle $\cos^2\theta + \sin^2\theta = 1$ equals $A(\theta) = \theta/2$. Considering the unit hyperbola $\cosh^2\theta - \sinh^2\theta = 1$, find the area enclosed between: (i) the line from the origin to a point P on the hyperbola, (ii) the line from the origin to the vertex, and (iii) the curve from the vertex to P. (Answer: $\theta/2$)

7. Prove that the arc length of a hyperbolic segment is
$$L = a \int d\theta \sqrt{e^2 \cosh^2\theta - 1},$$
which has no closed solution in general. This is another elliptic integral.

8. The axis of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is tilted by 45 degrees using the origin as a pivot so that the new axis lies along the line $y = x$. (The axis cuts through both focus points and the rotated hyperbola lives strictly in the first and third quadrants.) Prove that the equation of the tilted hyperbola is
$$y = x\frac{B}{A} \pm \frac{ab}{A}\sqrt{4x^2 - 2A},$$
where $A = a^2 - b^2$ and $B = a^2 + b^2$.

9. The result derived in the previous problem blows up for $a = b$ because you encounter division by zero. Supposing $x^2 - y^2 = 2$ with $a = b = 1$, what hyperbola *should* you get after the 45 degree rotation? Prove it. (Answer: $y = 1/x$)

10. The hyperbola $y = 1/x$, defined on the interval $x \geq 1$, is rotated about the x-axis so a semi-infinite three-dimensional 'horn' is made. Show that the total surface area of the shape is infinite. Determine the volume inside the horn (not infinite). (Answer: $A = 2\pi \int_1^\infty y(x) \sqrt{1 + (dy/dx)^2} dx$, $V = \pi$)