Complex Numbers

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1 Introduction

Complex numbers are a brilliant extension of the 'ordinary' real numbers that have enormous consequence in mathematics and physics. Begin by defining the complex number z as an ordered pair of real numbers A and B:

$$z = (A, B) \tag{1}$$

Next, declare that every complex number z has a dark twin called the *complex conjugate*, denoted $\overline{z}$, such that

$$\overline{z} = (A, -B)\ . \tag{2}$$

If we flip the sign on both A and B, the complex number and its conjugate gain a minus sign, as

$$-z = (-A, -B) \qquad\qquad -\overline{z} = (-A, B)$$

is equivalent to multiplying through by negative one. This readily generalizes to scalar multiplication by any real number h:

$$hz = (hA, hB) \tag{3}$$

2 Complex Algebra

2.1 Addition

A natural way to combine complex numbers $z_1 + z_2$ is to add similar components according to

$$\begin{aligned} z_1 + z_2 &= (A_1, B_1) + (A_2, B_2) \\ &= (A_1 + A_2, B_1 + B_2)\ , \end{aligned} \tag{4}$$

where clearly such an addition rule obeys the commutation property

$$z_1 + z_2 = z_2 + z_1\ . \tag{5}$$

Problem 1

Derive the addition rule for complex conjugates:

$$\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2} \tag{6}$$

Solution 1

$$\overline{z_1} + \overline{z_2} = (A_1, -B_1) + (A_2, -B_2) = (A_1 + A_2, -B_1 - B_2)$$

Problem 2

Use the above definitions to solve for $(A, 0)$ and $(0, B)$:

$$(A, 0) = \frac{z + \overline{z}}{2} \qquad\qquad (0, B) = \frac{z - \overline{z}}{2} \tag{7}$$

Solution 2

$$\begin{aligned} z + \overline{z} &= (A, B) + (A, -B) = (2A, 0) \\ z - \overline{z} &= (A, B) + (-A, B) = (0, 2B) \end{aligned}$$

2.2 Multiplication

To establish rules for multiplication, let us simply change the plus sign to the multiplication (dot) operator while requiring similar conjugate relationships:

$$\overline{z_1} \cdot \overline{z_2} = \overline{z_1 \cdot z_2} \tag{8}$$

$$z_1 \cdot z_2 = z_2 \cdot z_1 \tag{9}$$

Note these are *not* satisfied by the naive multiplication rule $(A_1 \cdot A_2, B_1 \cdot B_2)$. Instead, start from a more general statement

$$z_1 \cdot z_2 = (A_1, B_1) \cdot (A_2, B_2) = (q, r)$$

that invokes every order-two combination of $A_j B_k$. Specifically,

$$q = \alpha A_1 A_2 + \beta A_1 B_2 + \gamma B_1 A_2 + \delta B_1 B_2$$

$$r = \tilde{\alpha} A_1 A_2 + \tilde{\beta} A_1 B_2 + \tilde{\gamma} B_1 A_2 + \tilde{\delta} B_1 B_2 \ ,$$

where the Greek characters are unknown constants having values 0, 1, or -1.

Problem 3

Use only the information on hand to determine all unknowns and unveil the proper complex multiplication formula:

$$z_1 \cdot z_2 = (A_1 A_2 - B_1 B_2, A_1 B_2 + B_1 A_2) \tag{10}$$

Solution 3

Taking the complex conjugate of $z_1 \cdot z_2$, all signs on B-terms reverse. The sign on q remains the same, and meanwhile the sign on r reverses. This can only be true if

$$\beta = \gamma = \tilde{\alpha} = \tilde{\delta} = 0 \ ,$$

and thus the updated q and r read

$$q = \alpha A_1 A_2 + \delta B_1 B_2 \qquad\qquad r = \tilde{\beta} A_1 B_2 + \tilde{\gamma} B_1 A_2 \ .$$

Next, the commutation relation (9) tells us that swapping index 1 for index 2 should leave the result unchanged. We learn nothing about the q-equation this way, but in the r-equation we see that $\tilde{\beta}$ and $\tilde{\gamma}$ must agree in sign and are thus equal:

$$\tilde{\beta} = \tilde{\gamma}$$

To get a grip on α and δ, consider a 'confused' complex number that swaps A with B. This leaves the r-equation unchanged, so the change must occur in the q-equation. We argue that multiplication involving the confused complex number mustn't yield the correct result, so α and δ must disagree in sign:

$$\alpha = -\delta$$

The absolute sign on α is evident from the pure case with $B_{1,2} = 0$ and $r = 0$. This corresponds to 'ordinary' multiplication, where q becomes $\alpha A_1 A_2$, implying $\alpha = 1$ to keep the sign positive.

To determine the sign on $\tilde{\beta}$, let one of the $B_i = 0$ and keep the other positive. This becomes the case for scalar multiplication, and we find $\tilde{\beta}$ must equal 1:

$$(A_j, 0) \cdot (A_k, B_k) = \left(A_j A_k, \tilde{\beta} A_j B_k\right) = A_j \left(A_k, \tilde{\beta} B_k\right)$$

Problem 4

Prove the *associative property*:

$$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3) \tag{11}$$

Solution 4

$$\begin{aligned}
(z_1 \cdot z_2) \cdot z_3 &= (A_1 A_2 - B_1 B_2, A_1 B_2 + B_1 A_2) \cdot (A_3, B_3) \\
&= (A_1 A_2 A_3 - B_1 B_2 A_3 - A_1 B_2 B_3 - B_1 A_2 B_3, \\
&\qquad A_1 A_2 B_3 - B_1 B_2 B_3 + A_1 B_2 A_3 + B_1 A_2 A_3) \\
&= (A_1 (A_2 A_3 - B_2 B_3) - B_1 (B_2 A_3 + A_2 B_3), \\
&\qquad A_1 (A_2 B_3 + B_2 A_3) + B_1 (B_2 B_3 - A_2 A_3)) \\
&= (A_1, B_1) \cdot (A_2 A_3 - B_2 B_3, A_2 B_2 + B_2 A_3) \\
&= z_1 \cdot (z_2 \cdot z_3)
\end{aligned}$$

Problem 5

Prove the *distributive property*:

$$z_1 \cdot (z_2 + z_3) = (z_1 \cdot z_2) + (z_1 \cdot z_3) \tag{12}$$

Solution 5

$$\begin{aligned}
z_1 \cdot (z_2 + z_3) &= (A_1, B_1) \cdot (A_2 + A_3, B_2 + B_3) \\
&= (A_1 A_2 + A_1 A_3 - B_1 B_2 - B_1 B_3, \\
&\qquad A_1 B_2 + A_1 B_3 + A_2 B_1 + A_3 B_1) \\
&= (A_1 A_2 - B_1 B_2, A_1 B_2 + A_2 B_1) + \\
&\qquad (A_1 A_3 - B_1 B_3, A_1 B_3 + A_3 B_1) \\
&= (z_1 \cdot z_2) + (z_1 \cdot z_3)
\end{aligned}$$

Problem 6

Show that the *magnitude*

$$|z| = \sqrt{z \cdot \overline{z}} \tag{13}$$

of a complex number is analogous to the hypotenuse of a right triangle and has no complex component.

Solution 6

$$z_1 \cdot \overline{z}_1 = (A, B) \cdot (A, -B) = \left(A^2 + B^2, 0\right) = A^2 + B^2$$

Problem 7

The complex number z_I that leaves the other multiplicand unchanged is called he *identity*. Show this is simply the real number 1.

Solution 7

$$z_I \cdot z = (I_1, I_2) \cdot (A, B) = (I_1 A - I_2 B, I_1 B + I_2 A) = (A, B)$$

The only possible choice for z_I has $I_1 = 1$ and $I_2 = 0$, or $z_I = (1, 0) = 1$.

Problem 8

Prove there is no natural analog to geometric orthogonality for complex numbers by finding the only number z_2 that can be multiplied into nonzero z_1 to yield $(0, 0)$ as the result.

Solution 8

$$z_1 \cdot z_2 = 0 \qquad \rightarrow \qquad A_1 A_2 - B_1 B_2 = 0 \qquad A_1 B_2 + A_2 B_1 = 0$$

$$z_1 \cdot \overline{z}_2 = 0 \qquad \rightarrow \qquad A_1 A_2 + B_1 B_2 = 0 \qquad A_1 B_2 - A_2 B_1 = 0$$

In both cases, find $A_k^2 + B_k^2 = 0$, corresponding to a $A_k = B_k = 0$.

2.3 Division

Complex division is actually multiplication with style. Using properties of the complex conjugate, we derive the division formula for complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{z_1 \cdot \overline{z_2}}{|z_2|^2} \tag{14}$$

Complex Plane

Consider two statements that arise from the complex multiplication formula:

$$(1, 0) \cdot (A, B) = (A, B) \qquad\qquad (0, 1) \cdot (A, B) = (-B, A)$$

The first case corresponds to multiplying by the identity, whereas the second case looks suspiciously like a 90-degree rotation of a vector (A, B). Perhaps $(1, 0)$ and $(0, 1)$ are somehow analogous to basis vectors that span a plane in two dimensions.

In such a geometric analogy, a complex number $z = (A, B)$ that is 'rotated' into another position z' must be

$$z' = h\,(1, 0) \cdot (A, B) + k\,(0, 1) \cdot (A, B) = (h, k) \cdot (A, B)\ .$$

Problem 9

In order to preserve the magnitude of $|z| = |z'|$, show that h and k in the rotated complex number z' must obey

$$h^2 + k^2 = 1 \ .$$

Solution 9

$$|z'| = z' \cdot \overline{z'} = \left(h^2 + k^2\right)(z \cdot \overline{z}) = \left(h^2 + k^2\right)|z|$$

Evidently the available h and k live on the unit circle, so we re-parameterize via

$$h = \cos\theta \qquad\qquad k = \sin\theta \ ,$$

where θ is known as the *phase* of the number. This reduces the number of parameters floating around to three: A, B, and θ - which is one too many. Since θ alone can track the orientation of a rotated complex number, it follows that the magnitude is wrapped up in A and B, one of which can be eliminated. Set $B = 0$ and rename A to $|z| = r$, as in 'radius', availing a geometric representation of a complex number:

$$z = (|z|, 0) \cdot (\cos\theta, \sin\theta) = (r\cos\theta, r\sin\theta) \tag{15}$$

It's a lovely coincidence that equation (15) is exactly like the position vector in $2D$ polar coordinates. In the regime we develop here, the space of all complex numbers is called the *complex plane*.

2.4 Connection to Vectors

Similar to vectors, complex numbers contain the notion of magnitude, direction, transformation, and so on. Begin with the geometric representation (15) of a complex number z_1 and solve for $\cos\theta_1$ and $\sin\theta_1$:

$$\cos\theta_1 = \frac{A_1}{|z_1|} \qquad\qquad \sin\theta_1 = \frac{B_1}{|z_1|}$$

Next, break apart a similarly-represented complex number z_2 and multiply similar trig components via

$$\cos\theta_1\cos\theta_2 = \frac{A_1 A_2}{|z_1||z_2|} \qquad\qquad \sin\theta_1\sin\theta_2 = \frac{B_1 B_2}{|z_1||z_2|} \ ,$$

and add the two equations to get

$$\cos(\theta_2 - \theta_1) = \frac{A_1 A_2 + B_1 B_2}{|z_1||z_2|} \ ,$$

which is precisely the definition of the traditional dot (vector) product:

$$z_1 \circ z_2 = |z_1||z_2|\cos(\theta_2 - \theta_1) = A_1 A_2 + B_1 B_2$$

Problem 10

Modify the previous calculation to derive an expression for the cross product:

$$z_1 \times z_2 = A_1 B_2 - B_1 A_2$$

Solution 10

$$\cos\theta_1 \sin\theta_2 = \frac{A_1 B_2}{|z_1|\,|z_2|} \qquad\qquad \sin\theta_1 \cos\theta_2 = \frac{B_1 A_2}{|z_1|\,|z_2|}$$

Problem 11

Use the above relations to show that

$$\overline{z_1} \cdot z_2 = (z_1 \circ z_2, z_1 \times z_2)$$

Solution 11

Let $B_1 \to -B_1$.

Problem 12

Derive the following expressions for $z_1 \circ z_2$ and $z_1 \times z_2$ in terms of complex products:

$$(z_1 \circ z_2, 0) = \frac{1}{2}\left(\overline{z_1} \cdot z_2 + z_1 \cdot \overline{z_2}\right) \tag{16}$$

$$(0, z_1 \times z_2) = \frac{1}{2}\left(\overline{z_1} \cdot z_2 - z_1 \cdot \overline{z_2}\right) \tag{17}$$

Solution 12

Use equation (7).

2.5 Very Complex Numbers

We may also address cases when the components of a complex number z are themselves complex numbers. Suppose we're handed the 'very complex' object $((A, C), B)$. This can be boiled down to an ordinary complex number over several steps. First, we separate (A, C) from B to get

$$((A, C), B) = ((A, C), 0) + (0, B) \ .$$

Simplify further to write

$$\begin{aligned} ((A, C), B) &= (A, C) + (0, B) \\ &= (A, C + B) \ , \end{aligned} \tag{18}$$

and we're done. Evidently $((A, C), B)$ simplifies by adding the C-component to B.

Simplifying $(A, (B, C))$ is slightly harder, involving a 90-degree rotation. From start to finish, we discover:

$$\begin{aligned} (A, (B, C)) &= (A, 0) + (0, (B, C)) \\ &= (A, 0) + (0, 1) \cdot ((B, C), 0) \\ &= (A, 0) + (0, 1) \cdot (B, C) \\ &= (A, 0) + (-C, B) \\ &= (A - C, B) \end{aligned} \tag{19}$$

That is, a second-second-component C gains a minus sign and joins the first component. It follows that equations (18) and (19) can be applied recursively to reduce any 'very complex number' into an ordinary one.

Equations (18) and (19), can be stacked together to reveal

$$((A, B), (C, D)) = (A - D, B + C) \ ,$$

which relates the 'very complex' notation to complex addition, giving

$$z_1 + z_2 = (A, B) + (C, D) = ((A, B), (D, -C)) \ .$$

A similar bridge exits to the complex multiplication formula. If we let

$$a = A_1 A_2 \qquad b = A_1 B_2 \qquad c = B_1 A_2 \qquad d = B_1 B_2 \ ,$$

then

$$\begin{aligned} z_1 \cdot z_2 &= (A_1 A_2 - B_1 B_2, A_1 B_2 + B_1 A_2) \\ &= (a - d, b + c) \\ &= ((a, b), (c, d)) \ , \end{aligned}$$

showing the connection. However, let's keep simplifying to bring the notation full circle:

$$\begin{aligned} z_1 \cdot z_2 &= ((a, b), (c, d)) \\ &= (A_1 (A_2, B_2), B_1 (A_2, B_2)) \\ &= (A_1, B_1) \cdot (A_2, B_2) \end{aligned}$$

3 Euler's Formula

Begin with the geometric interpretation of a complex number

$$z = (r \cos\theta, r \sin\theta) \ ,$$

and compute the differential dz. From calculus, we know

$$dz = \frac{\partial z}{\partial r} \, dr + \frac{\partial z}{\partial \theta} \, d\theta \ ,$$

where the ∂ symbol denotes a partial derivative. Evaluating this, we find

$$dz = (\cos\theta, \sin\theta) \, dr + (-r \sin\theta, r \cos\theta) \, d\theta \ .$$

Note the dr and $d\theta$ terms are shorthand for $(dr, 0)$ and $(d\theta, 0)$.

Use the complex division formula (14) to divide the above by z. Doing all steps correctly, you should get

$$\frac{dz}{z} = (1/r, 0) \, dr + (0, 1) \, d\theta = \left(\frac{dr}{r}, d\theta \right) \ ,$$

which can be integrated directly to get

$$\ln z = (\ln r, 0) + (0, \theta) \qquad\qquad e^{\ln z} = e^{(\ln r, 0)} e^{(0,\theta)} \ ,$$

and simplifying once more:

$$z = (r, 0) \cdot e^{(0,\theta)} \tag{20}$$

Comparing this to equation (15), we can pick out the famous Euler's equation:

$$e^{(0,\theta)} = (\cos\theta, \sin\theta) \tag{21}$$

3.1 Notation

We now update the 'parentheses-with-comma' (A, B) notation to the streamlined $A + iB$ notation. Consider two complex numbers $z_1 = (A_1, B_1)$ and $z_2 = (A_2, B_2)$, and postulate the 'comma-free' version

$$z_1 = j_1 A_1 + i_1 B_1 \qquad\qquad z_2 = j_2 A_2 + i_2 B_2 \ .$$

Adding z_1 and z_2, we find

$$z_1 + z_2 = j_1 A_1 + j_2 A_2 + i_1 B_1 + i_2 B_2 \ ,$$

where the addition formula (4) reminds that A_1 must add to A_2 to yield a real number, indicating that $j_1 = j_2 = j = 1$, so ignore all j's. Similarly i_1 and i_2 are equal to a common i, but it would be deeply wrong to also take $i = 1$, or any real number for that matter.

To proceed, write the product $z_1 \cdot z_2$ in the same notation, giving

$$z_1 \cdot z_2 = (A_1 + iB_1) \cdot (A_2 + iB_2) \ ,$$

which after a FOIL operation becomes

$$z_1 \cdot z_2 = A_1 A_2 + i^2 B_1 B_2 + i \left(A_1 B_2 + B_1 A_2\right) \ .$$

Comparing this to the established formula (10) for complex number multiplication, the two central roles of i become simultaneously apparent. Evidently, we find that

$$i^2 = -1 \ ,$$

which mans factors of i^2 land back in the real numbers. However, any remaining (or odd-power) i-terms are outside the real numbers and are called *imaginary*. In summary, we can shed the 'parentheses-with-comma' notation in favor of the 'real-and-imaginary' notation as:

$$z = (A, B) = A + iB = re^{i\theta} \qquad\qquad i^2 = -1 \tag{22}$$

Problem 13

Write Euler's formula in 'comma-free' notation, and explore the case $\theta = \pi$.

Solution 13

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad\qquad e^{i\pi} + 1 = 0$$

Problem 14

Derive expressions for complex multiplication and complex division in terms of r_j and θ_j instead of A_j and B_j.

Solution 14

$$z_1 \cdot z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 \left(\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right)$$
$$\frac{z_1}{z_2} = \frac{\left(r_1 e^{i\theta_1}\right)}{\left(r_2 e^{i\theta_2}\right)} = \frac{r_1}{r_2}\left(\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right)$$

3.2 Additional Topics

Acceleration

Problem 15

Take time derivatives of the complex number $z(t) = r(t)\, e^{i\theta(t)}$ to write the velocity and acceleration. Relate these to known results from polar coordinates.

Solution 15

Letting $dr/dt = \dot{r}$ and $\omega = d\theta/dt = \dot{\theta}$, we find:

$$z(t) = r(t)\, e^{i\theta(t)}$$
$$\frac{d}{dt} z(t) = \dot{r} e^{i\theta} + i e^{i\theta} r\omega$$
$$\frac{d^2}{dt^2} z(t) = e^{i\theta}\left(\ddot{r} - r\omega^2\right) + i e^{i\theta}\left(2\dot{r}\omega + r\dot{\omega}\right)$$

The $\dot{z}$- and $\ddot{z}$-terms are respectively equal to the velocity and acceleration vectors in polar coordinates if we interpret

$$e^{i\theta} \to \hat{r} \qquad\qquad i e^{i\theta} \to \hat{\phi}\,.$$

Hyperbolic Trigonometry

Combining Euler's formula with its complex conjugate, namely

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad\qquad e^{-i\theta} = \cos\theta - i\sin\theta\,,$$

we discover formulas for $\cos\theta$ and $\sin\theta$:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad\qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{23}$$

Re-parameterize $i\theta \to \theta$ to write down the hyperbolic cosine and the hyperbolic sine:

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \qquad\qquad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \tag{24}$$

Problem 16

Derive an equation relating $\cosh\theta$ to $\cos i\theta$, and similarly for sinh and sin.

Solution 16

$$\cos i\theta = \cosh\theta \qquad\qquad \sin i\theta = i\sinh\theta$$

4 Complex Functions

A complex function $w(z)$ of a single variable $z = x + iy$ has the structure

$$w(z) = u(x, y) + iv(x, y) \ ,$$

where u and v are the respective real and imaginary components of the function.

4.1 Partial Derivatives

Begin by calculating the differential in w while substituting dx and dy for their representations in terms of dz and $d\bar{z}$:

$$\begin{aligned} dw(z) &= dx\frac{\partial w}{\partial x} + dy\frac{\partial w}{\partial y} \\ &= \frac{1}{2}(dz + d\bar{z})\frac{\partial w}{\partial x} + \frac{1}{2i}(dz - d\bar{z})\frac{\partial w}{\partial y} \\ &= \frac{1}{2}dz\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) + \frac{1}{2}d\bar{z}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) \end{aligned}$$

Meanwhile, the differential of $w(z, \bar{z})$ reads

$$\delta w(z, \bar{z}) = \delta z\frac{\partial w}{\partial z} + \delta\bar{z}\frac{\partial w}{\partial \bar{z}} \ ,$$

where comparing the two equations provides a definition for $\partial w/\partial z$ and $\partial w/\partial\bar{z}$:

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \qquad\qquad \frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) \tag{25}$$

In place of x and y, we may cast complex functions in terms of r and θ, leading to the following derivative operators:

$$\begin{aligned} r\frac{\partial}{\partial r} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} \\ \frac{\partial}{\partial\theta} &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial \bar{z}} \\ z\frac{\partial}{\partial z} &= \frac{1}{2}\left(r\frac{\partial}{\partial r} - i\frac{\partial}{\partial\theta}\right) \\ \bar{z}\frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\theta}\right) \end{aligned}$$

Problem 17

Derive the relations above.

Solution 17

Use the chain rule on $w(r, \theta)$ and $w(z, \bar{z})$. Also use $x = r\cos\theta$ and $y = r\sin\theta$.

4.2 Total Derivative

While partial derivatives of complex functions are straightforward, the total derivative is trouble. Proceeding in a calculus-101 analogy, we write

$$\frac{dw(z, \bar{z})}{dz} = \lim_{\Delta z \to 0} \frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}} \frac{\Delta \bar{z}}{\Delta z},$$

where if $\Delta z = |\Delta z| e^{i\theta}$, then the ratio $\Delta\bar{z}/\Delta z$ becomes $e^{-2i\theta}$, which can have *any* phase θ as $\Delta z \to 0$. The best thing we can do about the total derivative is to restrict w to have no explicit $\bar{z}$-dependence, eliminating the second term altogether. We therefore take the following two equations as criteria of the total derivative:

$$\frac{dw}{dz} = \frac{\partial w}{dz} \qquad\qquad \frac{\partial w}{\partial \bar{z}} = 0 \tag{26}$$

4.3 Terminology

We have seen that a complex function's simultaneous dependence on the complex point z and its complex conjugate $\bar{z}$ can have an ambiguous total derivative. Such functions where z and $\bar{z}$ appear are denoted as $w(z, \bar{z})$, with the letter w reserved.

Many interesting complex functions only depend on z (with $\bar{z}$ absent), denoted $f(z)$. If the derivative df/dz exists then the function is called *analytic*. Points where df/dz does not exist are called *singular*, where isolated singular points are called *poles*. An *entire* function has no singular points in its domain.

4.4 Cauchy-Riemann Conditions

Complex functions with no explicit $\bar{z}$-dependence follow an analogy from vector calculus. Starting with $\partial w/\partial\bar{z} = 0$, compare with equation (25) to land at the famed *Cauchy-Riemann conditions*:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \qquad\qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{27}$$

For a function

$$f(z) = u(x, y) + iv(x, y)$$

obeying the Cauchy-Riemann conditions, we first note that for two constants c_1 and c_2, the level curves

$$u(x, y) = c_1 \qquad\qquad v(x, y) = c_2$$

are orthogonal, as

$$\begin{aligned}
\nabla u \cdot \nabla v &= \left(\frac{\partial u}{\partial x} \,\hat{x} + \frac{\partial u}{\partial y} \,\hat{y} \right) \cdot \left(\frac{\partial v}{\partial x} \,\hat{x} + \frac{\partial v}{\partial y} \,\hat{y} \right) \\
&= \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \\
&= \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} = 0 \,.
\end{aligned}$$

Moreover, the Laplacian of such a function obeying (27) always vanishes, as

$$\begin{aligned}
\nabla^2 f = \nabla \cdot \nabla f\,(x, y) &= \frac{\partial^2 u}{\partial_{xx}} + \frac{\partial^2 v}{\partial_{xx}} + \frac{\partial^2 u}{\partial_{yy}} + \frac{\partial^2 v}{\partial_{yy}} \\
&= -\frac{\partial^2 v}{\partial_{xy}} + \frac{\partial^2 u}{\partial_{xy}} + \frac{\partial^2 v}{\partial_{xy}} - \frac{\partial^2 u}{\partial_{xy}} = 0 \,,
\end{aligned}$$

or more strongly,

$$\nabla^2 u = 0 \qquad\qquad \nabla^2 v = 0 \,.$$

Conventionally, $u\,(x, y)$ and $v\,(x, y)$ are called *harmonic* functions.

<u>Problem 18</u>

Show why the following functions are considered analytic, and then calculate df/dz:

$$\begin{aligned}
f\,(z) &= x^2 + 2ixy - y^2 \\
f\,(z) &= i\theta + \ln r \\
f\,(z) &= r^\alpha e^{i\alpha\theta}
\end{aligned}$$

<u>Problem 19</u>

Show why the following functions are not analytic, and then calculate $\partial f/\partial z$:

$$\begin{aligned}
f\,(z) &= x^2 - y^2 \\
f\,(z) &= x^2 + iy^2 \\
f\,(z) &= r^2\,(\cos\theta + i \sin\theta)
\end{aligned}$$

<u>Solution 18</u>

$$df/dz = 2z \qquad\qquad df/dz = 1/z \qquad\qquad df/dz = \alpha z^{\alpha - 1}$$

<u>Solution 19</u>

$$\partial f/\partial z = z \qquad\qquad \partial f/\partial z = x + y \qquad\qquad \partial f/\partial z = 3r/2$$

4.5 Connection to Electromagnetism

Consider two real vector fields $\vec{A}$ and $\vec{B}$ defined in terms of harmonic functions $u(x, y)$, $v(x, y)$:

$$\vec{A} = u\,\hat{x} - v\,\hat{y} \qquad\qquad \vec{B} = v\,\hat{x} + u\,\hat{y} \tag{28}$$

The Cauchy-Riemann equations indicate that the divergence and curl of each field resemble Maxwell's equations in charge-free two-dimensional space:

$$\vec{\nabla} \cdot \vec{A} = 0 \qquad\qquad \vec{\nabla} \times \vec{A} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad\qquad \vec{\nabla} \times \vec{B} = 0$$

Problem 20

Find two problems from electromagnetism that are solved by the complex function:

$$f(z) = \frac{1}{z - z_0}$$

Solution 20

$$x - x_0 = \rho\cos\theta \qquad\qquad y - y_0 = \rho\sin\theta$$

$$\begin{aligned} f(z) &= \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2} - i\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} \\ &= \frac{1}{\rho}(\cos\theta - i\sin\theta) = u + iv \\ \vec{A} &= \frac{\cos\theta\,\hat{x} + \sin\theta\,\hat{y}}{\rho} = \frac{\hat{r}}{\rho} \\ &\propto \text{Electric field vector due to a line of charge.} \\ \vec{B} &= \frac{-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}}{\rho} = \frac{\hat{\phi}}{\rho} \\ &\propto \text{Magnetic field vector due to a line of current.} \end{aligned}$$

5 Contour Integrals

We have seen that a contour C in the complex plane is subject to the rules of ordinary calculus with special attention allocated to the derivative via the Cauchy-Riemann conditions. Now we develop the notion of integration in the complex plane. Consider a complex contour C that begins and ends at the respective points z_a and z_b. The integral of a function $f(z)$ over the contour C can be expressed in terms of a real parameter t via the chain rule:

$$\int_C f(z)\;dz = \int_{t_a}^{t_b} f(z(t))\,\frac{dz(t)}{dt}\;dt$$

Substituting $f(z) = u(x, y) + iv(x, y)$ and noting that $z' = x'(t) + iy'(t)$, we find, using the same notation that gave us equation (28):

$$\int_C f(z)\;dz = \int_C \vec{A} \cdot d\vec{l} + i\int_C \vec{B} \cdot d\vec{l} \tag{29}$$

5.1 Cauchy's Integral Theorem

We may apply Stokes's theorem to transform the line integrals in equation (29) into area integrals in the complex plane

$$\oint_C f(z)\ dz = \int_\Omega \hat{z} \cdot \left(\vec{\nabla} \times \vec{A}\right)\ dx\ dy + i \int_\Omega \hat{z} \cdot \left(\vec{\nabla} \times \vec{B}\right)\ dx\ dy\ ,$$

which of course evaluates to *zero* when region Ω is enclosed by C. This proves the Cauchy Integral Theorem, formally stating that if a function $f(z)$ is analytic in a simply-connected region R, then the integral along a closed path C in R equals zero.

5.2 Defects

Singular points in the integration region that cause $f(z)$ to become non-analytic must be 'stepped around' to be excluded from the contour C. Consider the integral

$$I_n^{(0)} = \oint_C \frac{dz}{(z - z_0)^n}\ , \tag{30}$$

where n is an integer and z_0 is a singular point interior to C. Since the integration contour may be arbitrarily shaped, we may choose a unit circular path around the point z_0, running counter-clockwise by convention, with

$$z(\theta) = z_0 + e^{i\theta} \qquad\qquad z'(\theta) = ie^{i\theta}\ .$$

Using the delta function

$$\delta(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\alpha t}\ d\alpha\ ,$$

the above integral becomes

$$I_n^{(0)} = 2\pi i \delta(n-1)\ , \tag{31}$$

which is zero unless $n = 1$.

5.3 Cauchy Integral Formula

Generalizing the above, we consider the integral

$$I_n = \oint_C \frac{f(z)\ dz}{(z - z_0)^n} \tag{32}$$

about a circular path of arbitrarily-small radius

$$z(\theta) = z_0 + re^{i\theta} \qquad\qquad r \to 0\ .$$

The act of taking $r \to 0$ is equivalent to expanding $f(z_0)$ by Taylor series to discard high-order terms, provided that derivatives of $f(z)$ exist. Using the expansion

$$f(z) = \sum_{q=0}^{\infty} \frac{f^{(q)}(z_0)}{q!} (z - z_0)^q\ ,$$

we have

$$I_n = \sum_{q=0}^{\infty} \frac{f^{(q)}(z_0)}{q!} I_{n-q}^{(0)} = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0) \ ,$$

giving us the Cauchy integral formula:

$$\oint_C \frac{f(z)\ dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0) \tag{33}$$

Problem 21

Prove that the existence of all derivatives $f^{(n)}(z)$ is guaranteed by the Cauchy integral formula.

Solution 21

...

5.4 Analytic Continuation

Taylor series is convergent until the contour C touches a singular point z_0, where the *radius of convergence* corresponds to the largest contour C_0. In a process called *analytic continuation*, we may choose a point z_1 inside C_0 where Taylor expansion is valid, implying a new contour C_1 centered on z_1 with its own radius of convergence, in which another Taylor expansion for $f(z)$ applies. The non-overlapping part of C_1 is new 'territory' that the z_0-centered approximation doesn't cover. Iterating this process, we may cover the whole complex plane, as long as singular points (and regions) are stepped around.

It readily follows that a closed contour integral that encloses singularities is equal to the sum of elementary integrals around the singularities. If the region of analyticity is a multiply-connected surface due to singularities, $f(z)$ may be multi-valued.

5.5 Laurent Series

A generalization of the Taylor series that includes both positive and negative exponents is the *Laurent series*:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \tag{34}$$

Problem 22

Show that the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)\ dz}{(z-z_0)^{n+1}} \qquad n = 0, \pm 1, \pm 2, \dots \ , \tag{35}$$

where Γ is a contour topologically equivalent to C_0.

Solution 22

$$I_n = \oint_C \frac{f(z)\ dz}{(z-z_0)^n} = \oint_C \sum_{m=-\infty}^{\infty} \frac{a_m (z-z_0)^m\ dz}{(z-z_0)^n} = \sum_{m=-\infty}^{\infty} a_m \oint_C \frac{dz}{(z-z_0)^{n-m}}$$

$$I_n = \sum_{m=-\infty}^{\infty} a_m I_{n-m}^{(0)} = \sum_{m=-\infty}^{\infty} a_m 2\pi i \delta(n-m-1) \qquad \rightarrow m = n-1$$

$$I_n = a_{n-1} 2\pi i$$

$$I_{n+1} = a_n 2\pi i$$

Example: Annulus

Consider a function $f(z)$ that is analytic in the annulus

$$R_1 \leq |z-z_0| \leq R_0$$

centered on z_0. Begin by writing the $n=0$ case of the Cauchy integral formula

$$2\pi i\, f(z) = \oint_C \frac{f(\tilde{z})\ d\tilde{z}}{\tilde{z}-z} ,$$

where the function $f(z)$ is approximated by Laurent series (34), and contour C encloses z. Next, stretch C so as to wrap the inside of the annulus, with a tight 'bridge' of canceling paths that connect the enclosing radii. The resulting contours are C_1 and C_0 with opposing directions of integration. The contour integral along C_0 corresponding to R_0 was solved previously and generates the $n \geq 0$ terms:

$$a_n = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z)\ dz}{(z-z_0)^{n+1}} \qquad n \geq 0 .$$

Along contour C_1, the fraction $1/(\tilde{z}-z)$ may be expanded via geometric series

$$\frac{1}{\tilde{z}-z} = \frac{1}{\tilde{z}-z_0+z_0-z} = \frac{1}{z_0-z} \frac{1}{1+\frac{\tilde{z}-z_0}{z_0-z}} = \sum_{m=0}^{\infty} \frac{(z_0-\tilde{z})^m}{(z_0-z)^{m+1}} ,$$

which guarantees convergence as $|z-z_0| > |\tilde{z}-z_0| = R_1$. To discover the restriction on a_n along C_1, replace $\tilde{z}-z$ and $f(\tilde{z})$ in the integral formula as follows:

$$2\pi i\, f(z) = \oint_{C_1} \frac{f(\tilde{z})\ d\tilde{z}}{\tilde{z}-z} = \oint_{C_1} \sum_{n=-\infty}^{\infty} a_n (\tilde{z}-z_0)^n \sum_{m=0}^{\infty} \frac{(z_0-\tilde{z})^m}{(z_0-z)^{m+1}}\, d\tilde{z}$$

$$2\pi i\, f(z) = \sum_{n=-\infty}^{\infty} a_n \sum_{m=0}^{\infty} (z-z_0)^{-(m+1)} \oint_{C_1} (\tilde{z}-z_0)^{m+n}\ d\tilde{z}$$

$$\cancel{2\pi i}\, f(z) = \cancel{2\pi i} \sum_{n=-\infty}^{\infty} a_n \sum_{m=0}^{\infty} (z-z_0)^{-(m+1)} \delta(-(m+n)-1)$$

$$\rightarrow m+n = -1$$

$$f(z) = \sum_{n=-1}^{-\infty} a_n (z-z_0)^n$$

Evidently, only the negative n-terms have survived on contour C_1. We conclude that

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)\; dz}{(z-z_0)^{n+1}} \qquad\qquad n < 0\,.$$

6 Residue Calculus

Our study of contour integrals in the complex plane has yielded several useful results. First, the Cauchy integral theorem tells us that an analytic function $f(z)$ integrated along a closed contour C free of singularities always resolves to zero. Isolated singularities (poles) $z_0^{(p)}$ are handled by expanding $f(z)$ as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(p)} \left(z - z_0^{(p)}\right)^n \ ,$$

where the coefficients a_n are given by, in accordance with equation (35),

$$a_n^{(p)} = \frac{1}{2\pi i} \oint_{C_0^{(p)}} \frac{f(z)\; dz}{\left(z - z_0^{(p)}\right)^{n+1}} \qquad\qquad n = 0, \pm 1, \pm 2, \ldots\,.$$

Take the special case $n = -1$ to write

$$2\pi i\, a_{-1}^{(p)} = \oint_{C_0^{(p)}} f(z)\; dz\ ,$$

telling us that the integral of $f(z)$ around a pole is equal to the constant $2\pi i\, a_{-1}^{(p)}$. This process may repeat for each pole inside the contour C, resulting in the sum

$$2\pi i \sum_p a_{-1}^{(p)} = \oint_C f(z)\; dz\ .$$

This amazing result tells us that the problem of solving contour integrals may be reduced to finding the Laurent coefficients $a_{-1}^{(p)}$ at each pole $z_0^{(p)}$. The coefficient $a_{-1}^{(p)}$ is called the *residue* at $z_0^{(p)}$:

$$2\pi i \sum_p \text{Res}\left[f\left(z_0^{(p)}\right)\right] = \oint_C f(z)\; dz \tag{36}$$

6.1 Calculating Residue(s)

The *order* of a pole $z_0^{(p)}$ is the lowest (most negative) index of the Laurent series expansion of $f(z)$ around the pole, where a *simple* pole has a lowest index of -1. This means a pole z_0 of order m has corresponding Laurent series

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n\ .$$

Now we introduce the function $g(z)$ such that

$$g(z) = (z - z_0)^m f(z) = \sum_{q=0}^{\infty} a_{q-m} (z - z_0)^q \ , \tag{37}$$

which bumps z_0 to the numerator. Since the sum index begins at zero, $g(z)$ is simply a Taylor series, meaning

$$a_{q-m} = \frac{g^{(q)}(z_0)}{(q)!} \ ,$$

where the case $q - m = -1$ gives the residue of $f(z)$ at z_0:

$$\text{Res}\left[f(z_0)\right] = \frac{g^{(m-1)}(z_0)}{(m-1)!} \tag{38}$$

For functions containing only simple poles, the above reduces to

$$\text{Res}\left[f(z_0)\right] = g(z_0) \ . \tag{39}$$

<u>Problem 23</u>

For functions of the form

$$f(z) = \frac{p(z)}{q(z)} \ ,$$

containing a simple pole in the denominator, show that the residue resolves to

$$\text{Res}\left[f(z_0)\right] = \frac{p(z_0)}{q'(z_0)} \ . \tag{40}$$

<u>Solution 23</u>

$$\begin{aligned} q'(z) &= \frac{q(z) - q(z_0)}{z - z_0} \rightarrow \frac{z - z_0}{q(z)} = \frac{1}{q'(z)} - \frac{\cancel{q(z_0)}}{q'(z)\ q(z)} = \frac{1}{q'(z)} \\ g(z_0) &= \frac{p(z)}{q(z)}(z - z_0)\bigg|_{z=z_0} = \frac{p(z_0)}{q'(z_0)} \end{aligned}$$

We are now prepared to evaluate otherwise-impossible integrals in the real domain by closing the integration contour in the complex plane. Then, all of the work is condensed to calculating the residue of each enclosed pole in accordance with equation (36).

6.2 Polynomial Functions

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$$

whose domain is the real number line. If we connect $x = -\infty$ to $x = \infty$ with a semicircular arc, the resulting contour encloses the upper half of the complex plane. In light of singular points, the integral is

$$I = \oint_C \frac{dz}{(z-i)(z+i)} \,,$$

which encloses one simple pole $z_0 = i$. Using equation (39) or (40), we find

$$\text{Res}\left[f(i)\right] = \frac{1}{2i} \qquad \rightarrow \qquad I = \pi \,.$$

<u>Problem 24</u>

Evaluate:

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$$

<u>Solution 24</u>

$$I = \oint_C \frac{dz}{(z+i)(z-i)(z+2i)(z-2i)} \qquad \text{Contour contains two poles.}$$

$$g_1(z) = \frac{\cancel{(z-i)}}{\cancel{(z-i)}(z+i)(z^2+4)} \qquad g_2(z) = \frac{\cancel{(z-2i)}}{(z^2+1)\cancel{(z-2i)}(z+2i)}$$

$$I = 2\pi i\left(g_1(i) + g_2(2i)\right) = \frac{\pi}{6}$$

(Or use partial fractions.)

<u>Problem 25</u>

Evaluate:

$$I = \int_0^{\infty} \frac{x^2\,dx}{(x^2+4)(x^2+9)}$$

<u>Solution 25</u>

$$2I = \oint_C \frac{z^2\,dz}{(z+2i)(z-2i)(z+3i)(z-3i)} \qquad \text{Contour contains two poles.}$$

$$g_1(z) = \frac{z^2\cancel{(z-2i)}}{(z+2i)\cancel{(z-2i)}(z^2+9)} \qquad g_2(z) = \frac{z^2\cancel{(z-3i)}}{(z^2+4)(z+3i)\cancel{(z-3i)}}$$

$$I = \frac{2\pi i}{2}\left(g_1(2i) + g_2(3i)\right) = \frac{\pi}{10}$$

<u>Problem 26</u>

Evaluate:

$$I = \int_0^{\infty} \frac{dx}{\left(4x^2+1\right)^3}$$

<u>Solution 26</u>

$$2I = \oint_C \frac{dz}{4^3 \left(z + \frac{i}{2}\right)^3 \left(z - \frac{i}{2}\right)^3}$$

Contour contains one order-3 pole.

$$g(z) = \frac{\cancel{\left(z - \frac{i}{2}\right)^3}}{4^3 \left(z + \frac{i}{2}\right)^3 \cancel{\left(z - \frac{i}{2}\right)^3}}$$

$$I = \frac{2\pi i}{2} \left(\frac{1}{2} \frac{d^2}{dz^2} g(z) \bigg|_{z=i/2} \right) = \frac{3\pi}{32}$$

6.3 Jordan's Lemma for Fourier Transform

Starting with the integral

$$I = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx\,,$$

we carry the problem to the complex plane under three assumptions: (i) $k > 0$ and is a real number, (ii) $f(z)$ is analytic in the upper-half plane with the exception of simple poles, (iii) $\lim_{|z|\to\infty} f(z) = 0$. *Jordan's lemma* states that the integration path can be closed by an infinite semicircle in the upper-half plane. For th $k < 0$ case, the path would enclose the lower half-plane.

6.4 Sine and Cosine in Polynomial

For a real variable $a > 0$, the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2 + a^2}\, dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{ikx}\, dx}{x^2 + a^2} = \oint_C \frac{e^{ikz}\, dz}{z^2 + a^2}$$

is easily recast as a contour integral using Jordan's lemma. Using $g(z) = e^{ikz}/(z + ia)$, we quickly find $I = (\pi/a)\, e^{-ka}$. To generalize this, we have:

$$\int f(x) \cos(kx)\; dx = \mathrm{Re} \int f(x)\, e^{ikx}\, dx \tag{41}$$

$$\int f(x) \sin(kx)\; dx = \mathrm{Im} \int f(x)\, e^{ikx}\, dx \tag{42}$$

Problem 27

Evaluate:

$$I = \int_0^{\infty} \frac{\cos 2x}{9x^2 + 4}\, dx$$

Solution 27

$$I = \frac{1}{18}\, \mathrm{Re} \oint_C \frac{e^{2iz}\, dz}{\left(z + \frac{2i}{3}\right)\left(z - \frac{2i}{3}\right)} \qquad g(z) = \frac{e^{2iz}}{z + \frac{2i}{3}}$$

$$g(2i/3) = \frac{3}{4i}\, e^{-4/3} \qquad I = \frac{2\pi i}{18} \frac{3}{4i}\, e^{-4/3} = \frac{\pi}{12}\, e^{-4/3}$$

Problem 28

Evaluate:

$$I = \int_0^\infty \frac{\cos 2x}{\left(9x^2+4\right)^2}\, dx$$

Solution 28

$$I = \frac{1}{2 \cdot 9^2}\, \text{Re} \oint_C \frac{e^{2iz}\, dz}{\left(z+\frac{2i}{3}\right)^2 \left(z-\frac{2i}{3}\right)^2} \qquad g\left(z\right) = \frac{e^{2iz} \cancel{\left(z-\frac{2i}{3}\right)^2}}{\left(z+\frac{2i}{3}\right)^2 \cancel{\left(z-\frac{2i}{3}\right)^2}}$$

$$g^{(1)}\left(2i/3\right) = -i \left(\frac{14}{3} \cdot \frac{3^3}{4^3}\right) e^{-4/3} \qquad I = 2\pi i \left(g^{(1)}\left(2i/3\right)\right) = \frac{7\pi}{288}\, e^{-4/3}$$

Problem 29

Evaluate:

$$I = \int_{-\infty}^\infty \frac{\cos kx}{\left(x^2+a^2\right)^2}\, dx$$

Solution 29

$$I = \text{Re} \oint_C \frac{e^{ikz}\, dz}{\left(z^2+a^2\right)^2} \qquad g\left(z\right) = \frac{e^{ikz}}{\left(z+ia\right)^2}$$

$$g^{(1)}\left(ia\right) = \frac{i}{4\, e^{ka}} \frac{ka+1}{a^3} \qquad I = \frac{\pi}{2\, e^{ka}} \frac{ka+1}{a^3}$$

Problem 30

Evaluate:

$$I = \int_{-\infty}^\infty \frac{x \sin kx}{x^2+a^2}\, dx$$

Solution 30

$$I = \text{Im} \oint_C \frac{z\, e^{ikz}\, dz}{z^2+a^2} \qquad g\left(z\right) = \frac{z\, e^{ikz}}{z+ia}$$

$$g\left(ia\right) = \frac{1}{2} e^{-ka} \qquad I = \text{Im}\, \frac{2\pi i}{2\, e^{ka}} = \frac{\pi}{e^{ka}}$$

Problem 31

Evaluate:

$$I = \int_{-\infty}^\infty \frac{x \sin kx}{\left(x^2+a^2\right)^2}\, dx$$

Solution 31

$$I = -\frac{\partial}{\partial k} \int_{-\infty}^\infty \frac{\cos kx}{\left(x^2+a^2\right)^2}\, dx = -\frac{\partial}{\partial k} \left(\frac{\pi}{2\, e^{ka}} \frac{ka+1}{a^3}\right) = \frac{\pi k}{2a\, e^{ka}}$$

or

$$I = \text{Im} \oint_C \frac{z\, e^{ikz}\, dz}{\left(z^2+a^2\right)^2} = \text{Im}\, \frac{2\pi i k}{4a\, e^{ka}} = \frac{\pi k}{2a\, e^{ka}}$$

Problem 32

Evaluate:

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4x + 5}\, dx$$

Solution 32

$$I = \operatorname{Im} \oint_C \frac{z\, e^{ikz}\, dz}{(z+2+i)(z+2-i)} \qquad g(z) = \frac{z\, e^{ikz} \cancel{(z+2-i)}}{(z+2+i)\cancel{(z+2-i)}}$$

$$g(-2+i) = \frac{(-2+i)\ e^{i(-2+i)}}{2i} \qquad I = \frac{\pi}{e}\left(2\sin 2 + \cos 2\right)$$

6.5 Trigonometric Functions

Integrals of the form

$$I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

can be recast by expressing all θ-terms in terms of z and choosing an integration contour C_0 as the unit circle centered on $z = 0$. With help from equation (23), and also noting that $dz = ir\, e^{i\theta}\, d\theta$, the above becomes

$$I = -i \oint_{C_0} f\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{dz}{z}\,.$$

Problem 33

Evaluate:

$$I = \int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$$

Solution 33

$$I = -i \oint_{C_0} \frac{dz}{z} \frac{1}{13 + (5/2i)(z - \bar{z})} = \oint_{C_0} \frac{dz}{\frac{5}{2}\left(z + \frac{i}{5}\right)(z + 5i)}$$

$$g(z) = \frac{\cancel{\left(z + \frac{i}{5}\right)}}{\frac{5}{2}\cancel{\left(z + \frac{i}{5}\right)}(z + 5i)} \qquad g\left(\frac{-i}{5}\right) = \frac{-i}{12} \qquad I = 2\pi i \left(\frac{-i}{12}\right) = \frac{\pi}{6}$$

Problem 34

Evaluate:

$$I = \int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$$

Solution 34

$$I = -i \oint_{C_0} \frac{dz}{z} \frac{1}{5 + 2i(z - \bar{z})} = -\oint_{C_0} \frac{dz}{2\left(z - \frac{i}{2}\right)(z - 2i)}$$

$$g(z) = \frac{-1 \cdot \cancel{\left(z - \frac{i}{2}\right)}}{\cancel{\left(z - \frac{i}{2}\right)} 2(z - 2i)} \qquad g\left(\frac{i}{2}\right) = \frac{-i}{3} \qquad I = 2\pi i \left(\frac{-i}{3}\right) = \frac{2\pi}{3}$$

Problem 35

Evaluate:

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} \qquad\qquad a > |b|$$

Solution 35

$$I = -i \oint_{C_0} \frac{2\,dz}{2az + b\left(1 + z^2\right)} = -i \oint_{C_0} \frac{2\,dz}{b\left(z + \frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}\right)\left(z + \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}\right)}$$

$$z_0 = -\frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b} \qquad\qquad g\left(z\right) = \frac{2\cancel{\left(z - z_0\right)}}{b\cancel{\left(z + \frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}\right)}\left(z + \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}\right)}$$

$$g\left(z_0\right) = \frac{1}{\sqrt{a^2 - b^2}} \qquad\qquad I = 2\pi i\left(\frac{-i}{\sqrt{a^2 - b^2}}\right) = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Problem 36

Evaluate:

$$I = \int_0^{\pi} \frac{d\theta}{\left(a + \cos\theta\right)^2} \qquad\qquad a > 1$$

Solution 36

$$I = -i \oint_{C_0} \frac{2z\,dz}{\left(2az + 1 + z^2\right)^2} = -i \oint_{C_0} \frac{2z\,dz}{\left(z + a - \sqrt{a^2 - 1}\right)^2\left(z + a + \sqrt{a^2 - 1}\right)^2}$$

$$z_0^+ = -a + \sqrt{a^2 - 1} \qquad\qquad z_0^- = -a - \sqrt{a^2 - 1}$$

$$g\left(z\right) = \frac{2z\cancel{\left(z - z_0^+\right)^2}}{\cancel{\left(z - z_0^+\right)^2}\left(z - z_0^-\right)^2} \qquad\qquad g^{(1)}\left(z\right) = \frac{-2\left(z + z_0^-\right)}{\left(z - z_0^-\right)^3}$$

$$g^{(1)}\left(z_0^+\right) = \frac{a}{2\left(a^2 - 1\right)^{3/2}} \qquad\qquad I = 2\pi i\left(\frac{-ia}{2\left(a^2 - 1\right)^{3/2}}\right) = \frac{\pi a}{\left(a^2 - 1\right)^{3/2}}$$

6.6 Two-Contour Trick

The infinite complex plane (or a fraction of it) need not be enclosed by a semicircular contour. Rectangles are just as valid, which are an ideal application of the *two-contour trick*. This entails noticing when the integral of $f\left(z\right)$ on two enclosing contours C_1 and C_2 is the same up to a complex factor.

To illustrate, the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{1 + e^x} \qquad\qquad 0 > a > 1$$

may be rewritten

$$I = \int_{C_1} \frac{e^{az}\,dz}{1 + e^z}\,,$$

where contour C_1 is simply the real number line. Next, introduce a second contour C_2 that is shifted upward into the imaginary numbers but still parallel to the real line such that

$$z = x + 2\pi i \; .$$

Integrating 'backwards' along C_2, we have

$$-I \, e^{2i\pi a} = \int_{C_2} \frac{e^{az} \, dz}{1 + e^z} \; .$$

Of course, any contributions to the integral at $x = \pm\infty$ are zero, so we combine C_1 and C_2 to close the integration contour:

$$I \left(1 - e^{2i\pi a}\right) = \oint_C \frac{e^{az} \, dz}{1 + e^z}$$

<u>Problem 37</u>

Finish the calculation by solving the contour integral above.

<u>Solution 37</u>

$$z_0 = i\pi \qquad g(z) = \frac{(z - i\pi)\, e^{az}}{1 + e^z} \qquad g(z_0) \propto \frac{0}{0} \rightarrow \text{Need L'hopital.}$$

$$g(z_0) = \frac{e^{i\pi a}}{e^{i\pi}} \qquad I \left(1 - e^{2i\pi a}\right) = 2\pi i \frac{e^{i\pi a}}{e^{i\pi}} \qquad I = \frac{\pi}{\sin(\pi a)}$$

<u>Problem 38</u>

Use a pizza slice contour bounded by the positive real line and $z = r e^{2\pi i/n}$ (with vanishing crust at infinity) to evaluate

$$I = \int_0^\infty \frac{dx}{1 + x^n} \; .$$

<u>Solution 38</u>

$$I \left(1 - e^{2\pi i/n}\right) = \oint_C \frac{dz}{1 + z^n} \qquad z_0^n = -1 \rightarrow z_0 = (-1)^{-n} = e^{i\pi/n}$$

$$g(z) = \frac{\left(z - e^{i\pi/n}\right)}{1 + z^n} \qquad g(z_0) \propto \frac{0}{0} \rightarrow \text{Need L'hopital.}$$

$$g(z_0) = -\frac{e^{i\pi/n}}{n} \qquad I \left(1 - e^{2\pi i/n}\right) = -2\pi i \frac{e^{i\pi/n}}{n} \qquad I = \frac{\pi/n}{\sin(\pi/n)}$$

6.7 Regularization

Principal Value

Consider the *principal value* integral

$$I = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\ dx}{x - x_0}\ , \tag{43}$$

where by shifting $x \to z$, we assume that $f(z)$ is analytic except for a finite number of poles, and that $|f| \to 0$ on the upper (or lower) infinite semicircle in the complex plane.

Since the pole x_0 lies on the real axis, the integration contour cuts directly through x_0. This is handled by *regularization* of the denominator, which entails introducing a small factor $\delta > 0$ as

$$I = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\ dx}{x - x_0} = \lim_{\delta \to 0} \int_{-\infty}^{\infty} \frac{(x - x_0)\, f(x)\ dx}{(x - x_0)^2 + \delta^2} = \lim_{\delta \to 0} \oint_C \frac{(z - x_0)\, f(z)\ dz}{(z - x_0)^2 + \delta^2}\ .$$

After a little complex algebra, find

$$I = \lim_{\delta \to 0} \oint_C \frac{f(z)\ dz}{z - x_0 + i\delta} + \lim_{\delta \to 0} \oint_C i\delta \frac{f(z)\ dz}{(z - x_0 - i\delta)(z - x_0 + i\delta)}\ ,$$

which indicates one simple pole $z_0 = x_0 + i\delta$ inside the upper-half plane. The first integral in fact *excludes* the pole, so x_0 is skipped in subsequent residue calculations. (Use the δ-term as a reminder to skip x_0.) The second integral is solved by standard residue calculus, i.e., let $g(z) = f(z) / (z - x_0 + i\delta)$, resulting in $\pi i f(x_0)$.

Pulling the results together, we write

$$I^+ = \pi i f(x_0) + \lim_{\delta \to 0} \oint_C \frac{f(z)\ dz}{z - x_0 + i\delta}\ , \tag{44}$$

where if we started with $\delta < 0$ instead, the integration contour would flip to the lower-half plane, resulting in

$$I^- = -\pi i f(x_0) + \lim_{\delta \to 0} \oint_C \frac{f(z)\ dz}{z - x_0 - i\delta}\ .$$

In tighter notation (regardless of path or the sign of δ), one may write

$$I = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\ dx}{x - x_0} = \mathrm{P}\oint_C \frac{f(z)\ dz}{z - x_0}\ , \tag{45}$$

reminding us to *include* x_0 inside integration contour, but take the residue with a factor of 1/2.

<u>Problem 39</u>

Evaluate:

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

<u>Solution 39</u>

$$I = \mathrm{Im}\left(\pi i\, e^{i\cdot 0} + \lim_{\delta\to 0}\oint_C \frac{e^{iz}\, dz}{z + i\delta}\right) = \pi$$

Problem 40

Evaluate:

$$I = \mathrm{P}\int_{-\infty}^{\infty} \frac{\cos kx}{(x - x_0)\,(x^2 + 2)}\, dx \qquad\qquad J = \mathrm{P}\int_{-\infty}^{\infty} \frac{\sin kx}{(x - x_0)\,(x^2 + 2)}\, dx$$

Solution 40

$$\begin{aligned} K = \mathrm{P}\oint_C \frac{e^{ikz}}{(z - x_0)\,(z^2 + 2)}\, dz &= i\pi \frac{e^{ikx_0}}{x_0^2 + 2} + 2\pi i \left(\frac{e^{-k\sqrt{2}}}{\left(\sqrt{2}i - x_0\right)\left(2\sqrt{2}i\right)}\right) \\ &= \left(\frac{-\pi}{x_0^2 + 2}\left(\sin kx_0 + \frac{x_0\, e^{-\sqrt{2}k}}{\sqrt{2}}\right)\right) + i\left(\frac{\pi}{x_0^2 + 2}\left(\cos kx_0 - e^{-\sqrt{2}k}\right)\right) \\ &= I + iJ \end{aligned}$$

Dispersion Relations

One special case for $f(z)$ occurs when the upper-half plane contains no singularities, making the contour integral in equation (44) resolve to zero. By decomposing f into real and imaginary components u and v, respectively, we derive the *Kramers-Kroing* relations:

$$u(x_0) = \frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}\, dx \qquad\qquad v(x_0) = \frac{-1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}\, dx \tag{46}$$

6.8 Branch Cuts

Non-Integer Powers and Logarithms

Complex numbers with exponents and logarithms follow plainly from Euler's formula:

$$\begin{aligned} z^a &= r^a\, e^{ai\theta} \\ \ln z &= \ln r + i\theta \end{aligned}$$

Of course, the periodicity of θ leads to certain functions behaving non-smoothly as the line defined by $\theta = 0$ is crossed. For instance, the value of the logarithm

$$\ln z\,(r, 0) = \ln r \qquad\qquad \ln z\,(r, 2\pi) = \ln r + 2\pi i$$

at two equal points in the complex plane varies by (at least) $2\pi i$. Moreover, the square root $z^{1/2} = r^{1/2}\, e^{i\theta/2}$ is also multivalued, as

$$z^{1/2}\,(r, 0) = r^{1/2} \qquad\qquad z^{1/2}\,(r, 2\pi) = -r^{1/2}\,.$$

All of this is troublesome for contour integrals, so the line on which $f(z)$ is ill-behaved, called a *branch cut*, must be stepped around. To proceed generally we denote an initial phase θ_0 that defines a branch cut $z = r\, e^{i\theta_0}$, and then note

$$\theta_0 + 2\pi N \leq \theta < \theta_0 + 2\pi\,(N+1)$$

for an integer N, called the *branch*, that indexes multiples of 2π.

Problem 41

Show that the complex power and logarithm functions are analytic everywhere except for the branch θ_0.

Solution 41

$$\begin{aligned}
\bar{z}\frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right) \\
\bar{z}\frac{\partial}{\partial \bar{z}}\,(z^a) &= \frac{a}{2}\, e^{ia\theta}\,(r^a - r^a) = 0 \\
\bar{z}\frac{\partial}{\partial \bar{z}}\,(\ln z) &= \left(\frac{r}{r} + i^2\right) = 0
\end{aligned}$$

We next consider the integral

$$I = \int_0^\infty f(x)\, x^a\, dx\ ,$$

where $f(x)$ is well-behaved and non-singular on the real line, and the presence of x^a demands a branch but on $\theta_0 = 0$. To proceed we move the integral to the complex plane and go along three contours: (i) C_+, corresponding to $z = x + i\delta$ just above the real line, (ii) C_R, a nearly-full trip around the complex plane as $R \to \infty$, and (iii) C_-, coming from infinity back zero just below the real line with $z = x - i\delta$. Only the $C_\pm$ contours contribute to the integral, so in the limit $\delta \to 0$ we have:

$$I = \int_{C_+} f(z)\, z^a\, dz \qquad\qquad -\, e^{2i\pi a}\, I = \int_{C_-} f(z)\, z^a\, dz$$

Solving for I, the answer pops out:

$$\int_0^\infty f(x)\, x^a\, dx = \frac{1}{1 - e^{2i\pi a}} \oint_C f(z)\, z^a\, dz \tag{47}$$

Note that the integration contour surrounds the whole complex plane minus the branch cut. Don't forget to include all poles in residue calculations.

Problem 42

Evaluate:

$$I = \int_0^\infty \frac{x^a \, dx}{(1+x)^2}$$

<u>Solution 42</u>

$$\oint_C \frac{z^a \, dz}{(1+z)^2} = 2\pi i \sum_p \mathrm{Res}\left[f\left(z_0^{(p)}\right)\right] \qquad g(z) = \frac{z^a \cancel{(1+z)^2}}{\cancel{(1+z)^2}}$$

$$g^{(1)}(z) = a\, z^{a-1} \qquad g^{(-1)}(z) = a\, 1^{a-1}\, e^{i\pi a}\, e^{-i\pi} = -a\, e^{i\pi a}$$

$$I = \frac{-2\pi i a\, e^{i\pi a}}{1 - e^{2i\pi a}} = \frac{\pi a}{\sin(\pi a)}$$

The more general problem

$$I = \int_0^\infty f(x) \; dx$$

exploits a branch cut spectacularly. We proceed by considering a different integral $\tilde{I} = \int_0^\infty f(x) \ln(x) \; dx$ on the same contours C_+ and C_- used above, on the branch $0 \le \theta < 2\pi$. This gives us

$$\int_{C_+} f(z) \ln(z) \; dz = \tilde{I} \qquad \int_{C_-} f(z)\, z^a \, dz = -\tilde{I} - 2\pi i \int_0^\infty f(x) \; dx \,,$$

which sum together to perfectly cancel $\tilde{I}$, provided the usual assumptions that allow the integral with $R \to \infty$ limit to zero. Solving for the original I, we find:

$$\int_0^\infty f(x) \; dx = \frac{i}{2\pi} \int_C f(z) \ln(z) \; dz \tag{48}$$

<u>Problem 43</u>

Evaluate:

$$I = \int_0^\infty \frac{dx}{(x+2)(x+1)^2}$$

<u>Solution 43</u>

$$I = \frac{i}{2\pi} \cdot 2\pi i \left(\frac{d}{dz} \frac{\ln z}{z+2} \bigg|_{z=-1} + \frac{\ln z}{(z+1)^2} \bigg|_{z=-2} \right) = 1 - \ln 2$$