Central Forces

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Introduction

Central forces are ubiquitous in physics, arising in many situations when a particle is pushed or pulled at a distance from a point (presumably another particle). This is the main feature of gravitational and electrostatic interactions, and arises as a special case in more exotic systems.

1 Polar Coordinates

The Cartesian coordinate system with position vectors $\vec{r} = x\,\hat{x} + y\,\hat{y}$ is an awkward footing for central force analysis, whereas a system of *polar coordinates* is more natural. A set of polar coordinates that overlaps with Cartesian coordinates is expressed by

$$x = r\cos\theta \qquad\qquad y = r\sin\theta\,,$$

where r is the distance from the origin, and θ is an angular parameter where $\theta = 0$ coincides with the positive x-axis. The position vector $\vec{r}$ thus reads

$$\vec{r} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y}\,.$$

Unit Vectors

Divide the position vector $\vec{r}$ by its magnitude r to write the *radial unit vector* $\hat{r}$:

$$\hat{r} = \frac{\vec{r}}{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

The *angular unit vector* $\hat{\theta}$ is computed from the position vector as:

$$\hat{\theta} = \left|\frac{\partial\hat{r}}{\partial\theta}\right|^{-1}\left(\frac{\partial\hat{r}}{\partial\theta}\right) = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

Inverting the above equations to solve for $\hat{x}$ and $\hat{y}$ is straightforward in two dimensions.

$$\hat{x} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$\hat{y} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

Since the unit vectors $\hat{r}$, $\hat{\theta}$ depend on the coordinates in the $2D$ plane (much unlike the Cartesian unit vectors), their derivatives are nontrivial. Taking time derivatives, we find

$$\frac{d}{dt}\hat{r} = -\frac{d\theta}{dt}\sin\theta\,\hat{x} + \frac{d\theta}{dt}\cos\theta\,\hat{y} = \frac{d\theta}{dt}\,\hat{\theta}$$

$$\frac{d}{dt}\hat{\theta} = -\frac{d\theta}{dt}\cos\theta\,\hat{x} - \frac{d\theta}{dt}\sin\theta\,\hat{y} = -\frac{d\theta}{dt}\,\hat{r}$$

Velocity and Acceleration

While the position vector does not need a θ-component, its time derivative, namely the velocity, surely does:

$$\begin{aligned}\frac{d}{dt}\vec{r}\,(t) &= \frac{dr}{dt}\,\hat{r} + r\,\frac{d\hat{r}}{dt}\\ &= \frac{dr}{dt}\,\hat{r} + r\frac{d\theta}{dt}\,\hat{\theta}\end{aligned}$$

Further, the acceleration is straightforwad to calculate, although a bit messy:

$$\begin{aligned}\frac{d^2}{dt^2}\vec{r}\,(t) &= \frac{d^2 r}{dt^2}\,\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\,\hat{\theta} + r\frac{d^2\theta}{dt^2}\,\hat{\theta} + r\frac{d\theta}{dt}\frac{\hat{\theta}}{dt}\\ &= \frac{d^2 r}{dt^2}\,\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\,\hat{\theta} + r\frac{d^2\theta}{dt^2}\,\hat{\theta} + r\frac{d\theta}{dt}\frac{d\hat{\theta}}{dt}\\ &= \left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta}\end{aligned}$$

Lines and Arcs

In Cartesian coordinates, a differential line element is readily given by

$$d\vec{S} = dx\,\hat{x} + dy\,\hat{y}\ .$$

Using $x = r\cos\theta$, $y = r\sin\theta$, we find the differntial line element in polar coordinates becomes

$$\begin{aligned}d\vec{S} &= (dr\cos\theta - r\,d\theta\sin\theta)\,\hat{x} + (dr\sin\theta + r\,d\theta\cos\theta)\,\hat{y}\\ d\vec{S} &= dr\,\hat{r} + r\,d\theta\,\hat{\theta}\ .\end{aligned}$$

The absolute length of a line is calculated from

$$S = \int\sqrt{d\vec{S}\cdot d\vec{S}}\ ,$$

delivering two useful forms

$$S = \int_{x_i}^{x_f}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_{\theta_i}^{\theta_f}\sqrt{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2}\,r\,d\theta$$

Area Integrals

The total area swept out by the position vector $\vec{r}\,(\theta)$ is a sum of triangles having area $(1/2)\,r^2\,d\theta$ such that

$$A = \frac{1}{2}\int_{\theta_i}^{\theta_f}(r\,(\theta))^2\,d\theta\ .$$

2 Two-Body Problem

We proceed by studying two massive particles (labeled 1 and 2) interacting by an unspecified force. Newton's equations of motion read

$$m_1 \frac{d^2}{dt^2}\vec{r}_1 = \vec{F}_{12} \qquad\qquad m_2 \frac{d^2}{dt^2}\vec{r}_2 = \vec{F}_{21} \; ,$$

where position vectors $\vec{r}_{1,2}$ are measured from some origin, and the force F is equal and opposite between the particles.

Center of Mass

The center of mass of the two-body system is defined as

$$\vec{R}\,(t) = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \; ,$$

where $\vec{v}_{1,2}$ is equivalent to $d\vec{r}_{1,2}/dt$. Directly following from the equations of motion we find

$$\frac{d^2}{dt^2}\vec{R}\,(t) = 0$$

by direct calculation. That is, the center of mass moves at constant velocity $\vec{V}_0$.

Relative Displacement

The vector displacement

$$\vec{r}\,(t) = \vec{r}_1\,(t) - \vec{r}_2\,(t)$$

between the two particles avails a shortcut around writing separate differential equations for each position vector. In terms of $\vec{r}$ and $\vec{R}$, each position reads

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\,\vec{r} \qquad\qquad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\,\vec{r} \, .$$

Reduced Mass

To make use of the relative displacement vector, multiply the Newtonian equations of motion through by m_2, m_1 respectively to get

$$m_1 m_2 \frac{d^2}{dt^2}\vec{r}_1 = m_2 \vec{F}_{12} \qquad\qquad m_1 m_2 \frac{d^2}{dt^2}\vec{r}_2 = -m_1 \vec{F}_{12} \; ,$$

where $\vec{F}_{21} = -\vec{F}_{12}$ has been used, and then subtract the results to get

$$m_* \frac{d^2}{dt^2}\vec{r} = \vec{F}_{12} \qquad\qquad m_* = \frac{m_1 m_2}{m_1 + m_2} \; ,$$

where m_* is called the *reduced mass* of the system. The resulting differential equation for $\vec{r}\,(t)$ is equivalent to an equation of motion of a single particle of mass m_* with respect to an origin $\vec{r} = 0$.

External Potential Energy

From Newtonian mechanics, we know the external force vector $\vec{F}_{12}$ is the negative gradient of the *external potential* energy U_{12}, as in

$$\vec{F}_{12} = -\frac{\partial}{\partial \vec{r}} U_{12}(\vec{r}) = -\frac{\partial}{\partial r} U_{12}(r) \frac{\vec{r}}{r} \, .$$

Note that the vector $\vec{r}$ has been replaced by its magnitude r inside the argument of U_{12} from symmetry arguments, that is $U(\vec{r}) = U(r)$.

Total Energy

The total energy of a particle moving at velocity $\vec{v} = d\vec{r}/dt$ combines its kinetic energy with its external potential energy $U(r)$ to get

$$E = \frac{1}{2} m_* v^2 + U(r) \ .$$

It's readily shown that E is constant by taking a time derivative:

$$\begin{aligned} \frac{d}{dt} E &= m_* \, \vec{v} \cdot \frac{d\vec{v}}{dt} + \frac{d}{dt} U(r) \\ &= \frac{d\vec{r}}{dt} \cdot \left(m_* \frac{d\vec{v}}{dt} + \frac{\partial}{\partial \vec{r}} U(r) \right) \\ &= \vec{v} \cdot \cancel{\left(\vec{F}_{12} - \vec{F}_{12} \right)} \\ &= 0 \end{aligned}$$

Momentum

The *linear momentum* $\vec{p} = m_* \vec{v}$ is surely not conserved, however the *angular momentum about the orign* is a constant of motion. By definition, the angular momentum $\vec{L}$ reads

$$\vec{L} = m_* \vec{r} \times \vec{v} \, .$$

It's readily shown that L is constant by taking a time derivative

$$\begin{aligned} \frac{d}{dt} \vec{L} &= m_* \vec{r} \times \vec{v} \\ &= m_* \cancel{(\vec{v} \times \vec{v})} + m_* \, \vec{r} \times \vec{F}_{12} \\ &= 0 \, , \end{aligned}$$

which resolves to zero because $\vec{r}$ is parallel to $\vec{F}_{12}$. That is, there is no torque exerted in the system.

Using polar coordinates, the angular momentum of a test particle is

$$\begin{aligned} \vec{L} &= m_* \vec{r} \times \vec{v} \\ &= m_* \left(r \cos\theta \, \hat{x} + r \sin\theta \, \hat{y} \right) \times \frac{d\theta}{dt} \left(-r \sin\theta \, \hat{x} + r \cos\theta \, \hat{y} \right) \\ &= m_* r^2 \frac{d\theta}{dt} \, \hat{z} \, , \end{aligned}$$

giving us a tight expression for L in terms of the coordinates

$$L = m_* r^2 \frac{d\theta}{dt} \ .$$

Planar Motion

The angular momentum vector $\vec{L}$ is constant of motion, meaning it does not change sign or direction. It follows that $\vec{r}$ and $\vec{v}$ are always confined to the same plane perpendicular to $\vec{L}$, which is an important feature of central force motion: *all trajectories occur in a plane.*

Effective Potential Energy

Meanwhile, in terms of L, the energy of the particle becomes

$$E = \frac{1}{2} m_* \left(\frac{dr}{dt} \right)^2 + \frac{L^2}{2 m_* r^2} + U(r) \ ,$$

which is a first-order differential equation in r. The latter two terms naturally combine into the *effective potential energy*

$$U_{eff}(t) = U(r) + \frac{L^2}{2 m_* r^2} \ ,$$

where the combination $L^2/2m_* r^2$ is known as the *centrifugal potential energy*, the gradient of which is the *centrifugal force*. The energy is now the sum of distinct time-dependent and space-dependent parts:

$$E = \frac{1}{2} m_* \left(\frac{dr}{dt} \right)^2 + U_{eff}(r)$$

Effective Force

A time derivative of E reveals a one-dimensional version of Newton's second law:

$$m_* \frac{d^2}{dt^2} r(t) = F_{eff}(r) \qquad\qquad F_{eff}(r) = -\frac{d}{dr} U_{eff}(r)$$

3 One-Dimensional Motion

Two-body analysis considers a pair of interacting particles (m_1, m_2) as a single body of reduced mass m_* with an external potential energy. In deriving the analog to Newton's second law, we have introduced the notion of the *central potential*, i.e. $U(\vec{r}) = U(r)$. The total energy E and angular momentum $\vec{L}$ are constants of motion.

Before resuming a general study of central force motion, let us focus on the purely one-dimensional case. For a particle of mass m in a one-dimension with effective potential $U(x)$, we have Newtonian equations of motion

$$m \frac{d^2}{dt^2} x(t) = F(x) \qquad\qquad F(x) = -\frac{d}{dx} U(x) \ ,$$

implying the energy conservation equation

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x) \ .$$

In the general case, the solution to the above is given by

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

to get the equation of motion

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_i}^{x_f}\frac{dx}{\sqrt{E - U(x)}} \ .$$

Time-Reversal Symmetry

The $\pm$ symbol in the equation of motion indicates *time-reversal symmetry* of the problem. Typically in one-dimensional systems, the *solution* to the equation of motion exhibits such symmetry, a stronger constraint than what we have. Supposing $x(t)$ is a solution to the equation of motion, time-reversal symmetry implies that $x_1(t) = x(t_0 - t)$ is also a solution that differs from original $x(t)$ by an integration constant.

For most configurations, there exists at least one turning point t_* at which the velocity goes to zero. We exploit time-reversal symmetry to write an exact time-reversed-and-shifted equation

$$x_1(t) = x(2t_* - t) \ .$$

Next, we note from function- and derivative matching that

$$x_1(t_*) = x(t_*) \qquad\qquad \frac{d}{dt}x_1(x(t_*)) = -\frac{d}{dt}x(x(t_*)) = 0 \ ,$$

and so on for higher derivatives. We may then drop the 1-subscript to get

$$x(t) = x(2t_* - t) \ .$$

Shifting the above by t_*, the symmetric equation

$$x(t_* + t) = x(t_* - t)$$

emerges. In one dimension, the essence of time-reversal symmetry means that equations of motion are symmetric about turning points t_*.

Trapped Particle

Potential energy functions $U(x)$ that exhibit at least one local minimum can 'trap' a particle into an oscillatory pattern. Supposing x_i and x_f correspond to turning points in the motion, the oscillatory period is given by

$$T = \sqrt{2m}\int_{x_i}^{x_f}\frac{dx}{\sqrt{E - U(x)}} \ .$$

The quantity $E - U(x)$ is always positive except at the turning points, at which the speed of the particle is instantaneously zero.

Harmonic Oscillations

In the vicinity of a local energy minimum at x_*, the first- and second-derivatives of $U(x_*)$ are

$$\frac{d}{dx}U(x_*) = 0 \qquad\qquad \frac{d^2}{dx^2}U(x_*) = \lambda > 0\ ,$$

which allows $U(x)$ to be approximated by Taylor series:

$$U(x) \approx U(x_*) + \frac{1}{2}\lambda(x - x_*)^2$$

Applying Newton's second law, the corresponding equation of motion is

$$\frac{d^2}{dx^2}x(t) = -\frac{\lambda}{m}(x - x_*)\ ,$$

whose solution is known as the *harmonic oscillator*

$$x(t) = x_* + A\sin\left(\sqrt{\frac{\lambda}{m}}\,t - \phi_0\right)\ .$$

The amplitude of oscillation is A, and the initial phase is contained in ϕ_0.

Unstable Equilibrium

An equilibrium point x_* exists at any local maximum of $U(x)$, however motion around such a point is unstable (non-oscillatory). To show this, reverse the sign on λ to arrive at the differential equation

$$\frac{d^2}{dx^2}x(t) = \frac{\lambda}{m}(x - x_*)\ ,$$

generally solved by

$$x(t) = x_* + Ae^{\lambda t} + Be^{-\lambda t}\ .$$

That is, the particle is pulled away from x_* and rides $U(x)$ downhill.

4 Planar Orbits

We may apply the insights from one-dimensional analysis to the r-variable that occurs in the two-body problem. The differential equation governing $r(t)$, namely

$$E = \frac{1}{2}m_*\left(\frac{dr}{dt}\right)^2 + U_{eff}(r)\ ,$$

has easily-attainable solutions given by

$$t = \pm\sqrt{\frac{m_*}{2}}\int\frac{dr}{\sqrt{E - U_{eff}(r)}}\ .$$

Meanwhile, an equation for θ is given in terms of the (constant) angular momentum, namely $L = m_* r^2 \, d\theta/dt$, as in

$$\theta(t) = \theta_0 + \frac{L}{m_*} \int_0^t \frac{dt'}{r^2(t')} \, ,$$

which of course can be expressed in the r-domain instead:

$$\theta = \pm \frac{L}{\sqrt{2m_*}} \int \frac{dr/r^2}{\sqrt{E - U_{eff}(r)}}$$

The t- and θ-equations fully determine the geometry of the motion, called the *orbit.* The orbit (of a particle) is a special case of the kinematic trajectory in the sense that the initial conditions are not specified.

Apogee and Perigee

Supposing there exits a time t_* at which the radius r reaches a turning point (i.e. $dr/dt = 0$), the corresponding point (r_*, θ_*) in the plane is called the *apogee* if r is at a maximum, and the *perigee* if r is at a minimum. Solutions to the θ-equation occur in four explicit branches:

Apogee, $\theta > \theta_*$	$\theta = \theta_0 + \frac{L}{\sqrt{2m_*}} \int_{r_*}^{r} \frac{dr/r^2}{\sqrt{E - U_{eff}(r)}}$
Apogee, $\theta < \theta_*$	$\theta = \theta_0 - \frac{L}{\sqrt{2m_*}} \int_{r_*}^{r} \frac{dr/r^2}{\sqrt{E - U_{eff}(r)}}$
Perigee, $\theta > \theta_*$	$\theta = \theta_0 + \frac{L}{\sqrt{2m_*}} \int_{r}^{r_*} \frac{dr/r^2}{\sqrt{E - U_{eff}(r)}}$
Perigee, $\theta < \theta_*$	$\theta = \theta_0 - \frac{L}{\sqrt{2m_*}} \int_{r}^{r_*} \frac{dr/r^2}{\sqrt{E - U_{eff}(r)}}$

Emergent from the above is the reflection property about $\theta*$, namely

$$r(\theta_* - \theta) = r(\theta_* + \theta) \; .$$

Bounded Orbits

When the potential energy $U(x)$ contains a local minimum, a particle with a sufficiently low energy may become 'trapped' in the so-called *potential well.* Looking at the evolution of θ between two extreme points (perigee to apogee or vice-versa), we have

$$\theta = \frac{L}{\sqrt{2m_*}} \int_{r_p}^{r_a} \frac{dr/r^2}{\sqrt{E - U_{eff}(r)}} \, ,$$

which is some number that is not generally a rational fraction of π. That is, we find that orbits are bounded aren't necessarily repeated shapes, but may have a any (or an infinite) number of apogees and perigees. We will soon find there are two exceptions to this, where if the energy U has certain dependence on r, closed orbits are possible.

Circular Orbits

Any trapped particle can exhibit the special behavior of circular motion about the center, namely

$$r(t) = r_0 \qquad\qquad \theta(t) = \theta_0 + \frac{Lt}{m_* r_0^2} \; ,$$

in which case the effective potential energy is constant:

$$U_{eff}(r_0) = U(r_0) + \frac{L^2}{2m_* r_0^2} \qquad\qquad \frac{d}{dr}U_{eff}(r_0) = 0$$

The spatial derivative of the above tells us the force balance between the attraction and the centrifugal repulsion

$$\frac{d}{dr}U(r_0) = \frac{L^2}{m_* r_0^3} = \frac{m}{r_0}\left(r_0 \frac{d\theta}{dt}\right)^2 = \frac{m v_0^2}{r_0} \; ,$$

where $r \; d\theta/dt$ is the (constant) speed v_0 of the particle. The period of motion is the time required for θ to increase by 2π, easily written from our $\theta(t)$ solution:

$$T = \frac{2\pi m_* r_0^2}{L} = 2\pi\sqrt{\frac{m r_0}{U'(r_0)}} \qquad\qquad \frac{d}{dr}U(r_0) = U'(r_0)$$

5 Power Law Potentials

Most central potential energies follow a power law, namely

$$U(r) = -\frac{\gamma}{r^\alpha} \qquad\qquad U_{eff}(r) = -\frac{\gamma}{r^\alpha} + \frac{L^2}{2m_* r^2} \; ,$$

where γ and α are constants depending on the specific potential.

Circular Orbit Stability

For near-circular orbits, the effective potential energy $U_{eff}(r)$ with $\gamma > 0$ can be Taylor-expanded in the vicinity $r \approx r_0$ by

$$U_{eff}(r) \approx U_{eff}(r_0) + \cancel{\frac{d}{dr}U_{eff}(r)\Big|_{r_0}}(r - r_0) + \frac{d^2}{dr^2}U_{eff}(r)\Big|_{r_0}\frac{(r-r_0)^2}{2!} + ... \; ,$$

where the first-order term is zero by definition, and the second-order derivative term resolves to:

$$\begin{aligned}
\frac{d^2}{dr^2}U_{eff}(r_0) &= -\frac{\alpha(\alpha+1)\gamma}{r_0^{\alpha+2}} + \frac{3L^2}{m_* r_0^4} \\
&= -\frac{\alpha(\alpha+1)\gamma}{r_0^{\alpha+2}} + \frac{3}{r_0}\cancel{\left(\frac{d}{dr}U(r_0)\right)}\frac{\alpha\gamma}{r_0^{\alpha+1}} \\
&= \frac{\alpha\gamma}{r_0^{\alpha+2}}(2-\alpha) \\
&= \left(\frac{L^2}{m_* r_0^4}\right)(2-\alpha)
\end{aligned}$$

The orbital angular frequency ω of the circular orbit equals 2π over the period, namely

$$\omega_0 = \frac{2\pi}{T} = \frac{\cancel{2\pi} L}{\cancel{2\pi} m_* r_0^2} ,$$

simplifying the second-order derivative term further. As a result, we have

$$U_{eff}(r) \approx U_{eff}(r_0) + \frac{1}{2}\omega_0^2 (2-\alpha)(r-r_0)^2 ,$$

which has two nontrivial classes of behavior.

For all $\alpha > 2$, near-circular orbits are unstable, meaning particles with high enough energy will slip away to $r \to \infty$, whereas particles with sufficiently low energy or sufficiently low radius will inevitably collapse to $r = 0$. For $\alpha < 2$, the system corresponds to a one-dimensional harmonic oscillator in r, thus near-circular orbits are stable. The angular frequency in the r-variable is given by

$$\omega_r = \omega_0 \sqrt{2-\alpha} ,$$

implying that periodic closed orbits occur when $\sqrt{2-\alpha}$ is a rational number. Conveniently we'll see that the Coulomb and gravitational potentials ($\alpha = 1$), along with the harmonic oscillator ($\alpha = -2$) each produce closed orbits not limited to circles. The next closed orbit corresponds to $\alpha = -7$.

Harmonic Potential

Power law potential energies that restrict $\gamma > 0$ and $\alpha < 0$ always lead to bounded motion, not much different than the behavior of $U(r) \propto -1/r^\alpha$, in where the orbit is non-periodic (many apogees and perigees). One special case is the *harmonic potential*

$$U(r) = \gamma r^2 .$$

In three dimensions, the harmonic potential reads

$$U(r) = \gamma \left(x^2 + y^2 + z^2\right) ,$$

a separable differential equation. Since central force motion is always confined to a plane, let us choose to align the z-axis as perpendicular to the xy-plane of motion. This implies a pair of independent one-dimensional differential equations in x and y:

$$\frac{d^2}{dt^2}x(t) = -\omega^2 x(t) \qquad \frac{d^2}{dt^2}y(t) = -\omega^2 y(t) \qquad \omega = \sqrt{\frac{2\gamma}{m_*}}$$

General solutions to the above are trigonometric, namely

$$x(t) = A_x \cos(\omega t - \phi_x) \qquad y(t) = A_y \cos(\omega t - \phi_y) ,$$

where $A_{x,y}$ and $\phi_{x,y}$ are determined from initial conditions. We can do away with the ϕ_x-term by placing the particle at A_x at $t = 0$ and defining the x-axis to pass through that point. Then, the y-component of the position must be zero, telling us $\phi_y = \pi/2$. Finally, we find a closed equation for elliptical orbits with the origin at the center:

$$x(t) = A_x \cos(\omega t) \qquad y(t) = A_y \sin(\omega t) \qquad \frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1$$

Dimensionless Variables

The power law potential $U(r)$ does not imply any special cases in scale or energy. However, the effective potential $U_{eff}(r)$ embeds the notion of a minimal-energy circular orbit characterized by the angular momentum of the particle. It makes sense therefore to define three dimensionless variables

$$\rho = \frac{r(t)}{r_0}$$
$$\mathcal{E} = \frac{E}{|E_{min}|} = \frac{E}{|U_{eff}(r_0)|}$$
$$\tau = \frac{t}{T_{circ}} = t\frac{L}{2\pi m_* r_0^2}$$

where r_0 is the radius of the circular orbit corresponding to the lowest energy allowed, E_{min}. The condition for circular orbits $(d/dr)\,U_{eff}(r_0) = 0$ tells us how to relate r_0, $|E_{min}|$, L, and γ:

$$r_0^{(2-\alpha)} = \frac{L^2}{\alpha\gamma m_*}$$
$$|E_{min}| = \frac{2-\alpha}{\alpha}\frac{L^2}{2m_* r_0^2}$$

Using dimensionless variables, the equations of motion become

$$\theta = \pm\int_{\rho_i}^{\rho_f} \frac{d\rho/\rho^2}{\sqrt{(2-\alpha)\,\mathcal{E}/\alpha - 1/\rho^2 + (2/\alpha)\,/\rho^\alpha}}$$
$$\tau = \pm\left(\frac{1}{2\pi}\right)\int_{\rho_i}^{\rho_f} \frac{d\rho}{\sqrt{(2-\alpha)\,\mathcal{E}/\alpha - 1/\rho^2 + (2/\alpha)\,/\rho^\alpha}} .$$

In the above, that the square root term(s) resolve to zero at turning points $\rho = (\rho_a, \rho_p)$. Thus,

$$(2-\alpha)\,\mathcal{E}/\alpha - 1/\rho^2 + (2/\alpha)\,/\rho^\alpha = 0$$

can be used to determine all apogees and perigees.

6 Inverse-Square Attraction

The special case $(\alpha = 1)$, $(\gamma > 1)$

$$U(r) = -\frac{\gamma}{r}$$

is responsible for the gravitational force and the attractive Coulomb force. The circular orbit is natural to this potential such that

$$r_0 = \frac{L^2}{\gamma m_*} \qquad T_{circ} = \frac{2\pi L^3}{m_*\gamma^2} = \frac{2\pi m_* r_0^2}{L} \qquad |E_{min}| = \frac{m_*\gamma^2}{2L^2} .$$

The equation of motion for θ drastically simplifies to

$$\theta = \pm \int \frac{d\rho/\rho^2}{\sqrt{\mathcal{E} - 1/\rho^2 + 2/\rho}} \,,$$

which can be solved analytically by letting $\xi = 1/\rho$ and then $\beta = \xi - 1$ to get

$$\theta = \pm \int \frac{-d\xi}{\sqrt{-\left(\xi - 1\right)^2 + 1 + \mathcal{E}}} = \pm \int \frac{-d\beta}{\sqrt{-\beta^2 + 1 + \mathcal{E}}} \,.$$

Factor $1 + \mathcal{E}$ out of the square root and let $\gamma = \beta/\sqrt{1+\mathcal{E}}$:

$$\theta = \pm \int \frac{-d\beta/\sqrt{1+\mathcal{E}}}{\sqrt{-\beta^2/\left(1+\mathcal{E}\right) + 1}} = \pm \int \frac{-d\gamma}{\sqrt{1-\gamma^2}}$$

Next, let $\gamma = \cos\psi$ to get

$$\theta = \pm \int \frac{\cancel{\sin\psi}\, d\psi}{\cancel{\sin\psi}} = \pm\psi \,.$$

Undo each substitution all the way back to ρ and we finally get

$$\theta = \theta_0 \pm \arccos\left(\frac{1/\rho - 1}{\sqrt{1+\mathcal{E}}}\right) \,.$$

The integration constant is ignored as a mere rotation of the xy-plane. Solving the above for ρ, we find

$$\rho = \frac{1}{1 + \sqrt{1+\mathcal{E}}\cos\theta} \,,$$

the polar equation of a conic section with the origin at the right-hand focus.

The quantity

$$e = \sqrt{1+\mathcal{E}}$$

is identified as the *eccentricity* of the orbit. For all $e < 2$, the orbit corresponds to an ellipse. For $e > 2$, the motion is hyperbolic. The special case $e = 0$ corresponds to a parabola.

Runge-Lenz Vector

The fact that the $-\gamma/r$ potential supports closed stable orbits suggests a conserved quantity beyond energy and momentum. It turns out that the *Runge-Lenz vector* has been here all along, written as

$$\vec{Z} = \vec{v} \times \vec{L} - \gamma\, \hat{r} \,,$$

proven to be constant using a time derivative:

$$\begin{aligned}
\frac{d}{dt}\vec{Z} &= \frac{d\vec{v}}{dt} \times \vec{L} + \cancel{\vec{v} \times \frac{d\vec{L}}{dt}} - \gamma \frac{d\hat{r}}{dt} \\
&= -\frac{1}{m_*}\frac{\gamma}{r^2}\, \hat{r} \times \left(\vec{r} \times m_*\, \vec{v}\right) - \gamma \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \\
&= \gamma\left(\frac{-\left(\hat{r}\left(\hat{r}\cdot\vec{v}\right) - \vec{v}\left(\hat{r}\cdot\hat{r}\right)\right)}{r} - \frac{\vec{v}}{r} + \frac{\hat{r}}{r}\left(\hat{r}\cdot\vec{v}\right)\right) \\
&= 0
\end{aligned}$$

Being constant, we're free to evaluate $\vec{Z}$ anywhere on the orbit. Choosing a perigee at $\vec{r}_p = r_p\,\hat{x}$ where $\vec{v}_p \cdot \vec{r}_p = 0$, we find:

$$
\begin{aligned}
\vec{Z} &= \vec{v}_p \times (\vec{r}_p \times m_*\,\vec{v}_p) - \gamma\,\hat{r}_p \\
&= m_*\,\vec{r}_p\,(\vec{v}_p \cdot \vec{v}_p) - m_*\,\vec{v}\cancel{(\vec{v}_p \cdot \vec{r}_p)} - \gamma\,\hat{r}_p \\
&= \left(\frac{L^2}{m_* r_p} - \gamma\right)\hat{r}_p = \gamma\left(\frac{1}{\rho_p} - 1\right)\hat{x} \\
&= \gamma\left(1 + \sqrt{1+\mathcal{E}} - 1\right)\hat{x} \\
\vec{Z} &= \gamma e\,\hat{x}
\end{aligned}
$$

Interpreting $\vec{Z}$, it follows that all orbits in a $-\gamma/r$ potential contain at least one perigee, defining the x-axis, about which the orbit is symmetric. All other details of the orbit are contained in the eccentricity e.

Geometry

We verify all orbits to be conic sections by projecting $\vec{r}$ onto $\vec{Z}$:

$$
\begin{aligned}
\vec{r} \cdot \vec{Z} &= \vec{r} \cdot \left(\vec{v} \times \vec{L}\right) - \gamma \vec{r} \cdot \hat{r} \\
|\vec{r}|\left|\vec{Z}\right| \cos\theta &= \vec{L} \cdot (\vec{r} \times \vec{v}) - \gamma r \\
r\gamma e \cos\theta &= \frac{L^2}{m_*} - \gamma r \\
r &= \frac{r_0}{1 + e\cos\theta}
\end{aligned}
$$

We can relate a and b to r_0 using the $r\,(\theta)$ equation. Taking the difference between $r\,(0)$ and $r\,(\pi)$, we find

$$
a = \frac{1}{2}\left(\frac{r_0}{1+e} + \frac{r_0}{1-e}\right) = \frac{r_0}{1-e^2}\ .
$$

Solving for b isn't as simple. Proceed by calculating

$$
0 = \frac{d}{d\theta} y = \frac{d}{d\theta}\,(r \sin\theta)
$$

to find the angle θ_* that corresponds to the top of the ellipse:

$$
0 = \frac{d}{d\theta}\left(\frac{r_0 \sin\theta}{1 + e\cos\theta}\right) = r_0 \frac{e + \cos\theta}{(1 + e\cos\theta)^2} \qquad\qquad \cos\theta_* = -e
$$

At θ_*, it follows that

$$
r_* = \sqrt{e^2 a^2 + b^2} \qquad\qquad b = r_* \sin\theta_*\ ,
$$

leading to

$$
b = \frac{r_0}{\sqrt{1-e^2}}\ .
$$

For elliptical orbits, note that the semi-major axis a and semi-minor axis b relate to the eccentricity, and thus the physical qualities of the system, by

$$\frac{b}{a} = \sqrt{1-e^2} = \sqrt{-\mathcal{E}}\,,$$

reminding us that the energy of a closed orbit is always negative. Meanwhile, hyperbolic curves correspond to positive-energy orbits and obey

$$\frac{b}{a} = \sqrt{e^2-1} = \sqrt{\mathcal{E}}\,.$$

Dynamics

The time evolution of a particle in a $-\gamma/r$ potential is an analytically-solvable problem. Setting $\alpha = 1$ in the dimensionless equations of motion, we have, for the time component:

$$\begin{aligned}
\tau &= \pm\left(\frac{1}{2\pi}\right)\int_{\rho_i}^{\rho_f}\frac{\rho\,d\rho}{\sqrt{\mathcal{E}\rho^2+2\rho-1}} && \mathcal{E}\neq 0\\
&= \pm\left(\frac{1}{2\pi\sqrt{|\mathcal{E}|}}\right)\int\frac{\left(\beta/\sqrt{\mathcal{E}}-1/\mathcal{E}\right)d\beta}{\sqrt{\beta^2-(1/\mathcal{E}+1)}} && \beta = \sqrt{\mathcal{E}}\rho+\frac{1}{\sqrt{\mathcal{E}}}\\
&= \pm\left(\frac{1}{2\pi\sqrt{|\mathcal{E}|}}\right)\int\frac{\left(\beta/\sqrt{\mathcal{E}}-1/\mathcal{E}\right)d\left(\beta/\gamma\right)}{\sqrt{\left(\beta/\gamma\right)^2-1}} && \gamma = \sqrt{\frac{1}{\mathcal{E}}+1}\\
&= \pm\left(\frac{1}{2\pi\sqrt{|\mathcal{E}|}}\right)\int\frac{\left(\xi\gamma/\sqrt{\mathcal{E}}-1/\mathcal{E}\right)d\xi}{\sqrt{\xi^2-1}} && \xi = \frac{\beta}{\gamma}\\
&= \pm\left(\frac{1}{2\pi\sqrt{|\mathcal{E}|}}\right)\int\frac{\left(\lambda\xi-1/\mathcal{E}\right)d\xi}{\sqrt{\xi^2-1}} && \lambda = \frac{\gamma}{\sqrt{\mathcal{E}}} = \frac{\sqrt{1+\mathcal{E}}}{|\mathcal{E}|}
\end{aligned}$$

Depending on the sign of $\mathcal{E}$, the integral to solve is either of

$$\begin{aligned}
\tau &= \pm\left(\frac{1}{2\pi\sqrt{|\mathcal{E}|}}\right)\int\frac{\left(\lambda\xi-1/\mathcal{E}\right)d\xi}{\sqrt{1-\xi^2}} && \mathcal{E}<0\\
\tau &= \pm\left(\frac{1}{2\pi\sqrt{|\mathcal{E}|}}\right)\int\frac{\left(\lambda\xi-1/\mathcal{E}\right)d\xi}{\sqrt{\xi^2-1}} && \mathcal{E}>0\,.
\end{aligned}$$

To proceed, let

$$\begin{aligned}
\xi &= -\cos\psi && \mathcal{E}<0\\
\xi &= \cosh\psi && \mathcal{E}>0
\end{aligned}$$

such that the integrals become trivial to solve:

$$\tau = \pm \left(\frac{\psi - e \sin \psi}{2\pi \left|\mathcal{E}\right|^{3/2}} \right) \qquad \mathcal{E} < 0$$

$$\tau = \pm \left(\frac{e \sinh \psi - \psi}{2\pi \left|\mathcal{E}\right|^{3/2}} \right) \qquad \mathcal{E} > 0 \, .$$

Following the substitutions backward, we can also write equations for ρ:

$$\rho = \frac{1}{|\mathcal{E}|} \left(1 - e \cos \psi\right) \qquad \mathcal{E} < 0$$

$$\rho = \frac{1}{\mathcal{E}} \left(e \cosh \psi - 1\right) \qquad \mathcal{E} > 0$$

Note that the period of the orbit corresponds to the interval $0 \leq \psi < 2\pi$, which immediately tells us

$$\tau_{\text{period}} = 2\tau_{p \to a} = 2\tau_{0 \to \pi} = \left|\mathcal{E}\right|^{-3/2}$$

Zero-Energy Case

For the special case of parabolic orbits with $\mathcal{E} = 0$, the τ-integral reduces to

$$\tau = \pm \left(\frac{1}{2\pi} \right) \int \frac{\rho \, d\rho}{\sqrt{2\rho - 1}} \, ,$$

easily solved:

$$\tau = \pm \left(\frac{1}{6\pi} \right) \sqrt{2\rho - 1} \left(\rho + 1\right)$$

Inverse-Square Repulsion

The modified case $(\alpha = 1)$, $(\gamma > 1)$

$$U\left(r\right) = \frac{\gamma}{r}$$

is responsible for the repulsive Coulomb force. In analogy the the preceding analysis, it's straightforwardly shown that the ρ-equation gains an embedded minus sign

$$\rho = \frac{1}{-1 + \sqrt{1 + \mathcal{E} \cos \theta}} \, .$$

Meanwhile, the ψ-parameterization becomes:

$$\tau = \pm \left(\frac{e \sinh \psi + \psi}{2\pi \left|\mathcal{E}\right|^{3/2}} \right)$$

$$\rho = \frac{1}{\mathcal{E}} \left(e \cosh \psi + 1\right)$$

7 Kepler's Laws of Planetary Motion

Johannes Kepler (1571 - 1630) was a German natural philosopher. His eclectic career predates the Newtonian revolution, meaning there was no awareness of universal gravitation, the correct laws of motion, or calculus. Working from astronomical data accumulated by Tyco Brahe, who himself had a 30+ year career as a celestial observer, Kepler was able to discern three essential 'laws' that seemed to govern planetary motion.

Law of Ellipses (1609)

The orbit of each planet is an ellipse, with the sun located at a focus.

We verify Kepler's law of ellipses by recalling that the shape of a bounded orbit occurs as a conic section:

$$\rho = 1/\left(1 - e\cos\theta\right) \qquad r = 0 \text{ at left focus}$$
$$\rho = 1/\left(1 + e\cos\theta\right) \qquad r = 0 \text{ at right focus}$$

Law of Equal Areas (1609)

A line drawn between the sun and the planet sweeps out equal areas in equal times.

Just *how* Kepler came to ponder the law of equal areas is itself mysterious, because it *does* turn out that the radius vector sweeps out equal areas in equal times. To prove this, recall the integral that computes area in polar coordinates, namely

$$A\left(\theta_f - \theta_i\right) = \frac{1}{2}\int_{\theta_i}^{\theta_f} r^2 \, d\theta \, .$$

Also recall that the angular momentum of a planet in orbit is constant, as in

$$L = m_* r^2 \frac{d\theta}{dt} \, ,$$

which brings the area integral from θ to the time domain:

$$A\left(t_i - t_f\right) = \frac{L}{2m_*}\int_{t_i}^{t_f} dt$$

Setting the initial condition $A\left(t = 0\right) = 0$, the above resolves to

$$A\left(t\right) = \frac{Lt}{2m_*} \, ,$$

which embeds the mathematical statement of Kepler's law of equal areas:

$$\frac{d}{dt}A\left(t\right) = \frac{L}{2m_*}$$

Kepler was unaware that the right side of the result is $L/2m_*$, but surely knew it is constant for a given planet.

Harmonic Law (1618)

The square of the period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Years after his first two discoveries, Kepler discerned yet another relationship for linking the time scale of the orbit to its length scale. While Kepler only knew of the proportionality between the period and the semi-major axis, we can do better by finding the associated constant.

Starting with the $A(t)$ equation and letting $t = T$ for one full orbit, the left side is the area πab of the closed ellipse. Simplifying, we recover a familiar result $\tau_{\text{period}} = |\mathcal{E}|^{-3/2}$:

$$\begin{aligned}\pi ab &= \frac{L}{2m_*}T \\ \pi a^2 \frac{\sqrt{1-e^2}}{\sqrt{r_0}} &= \frac{\sqrt{\gamma m_*}}{2m_*}T \\ T &= \frac{2\pi}{\sqrt{\gamma/m_*}}a^{3/2}\end{aligned}$$

8 Newton's Law of Gravitation

Isaac Newton's approach to planetary motion started with the observation that planets follow elliptical orbits with the sun at a focus (not the center), from which he reverse-engineered the $-1/r^2$ signature of the gravitational force. While our notation is different than Newton's, let us 'stumble upon' the law of gravitation in a similar way.

Inverse-Square Attraction

Take the acceleration vector in polar coordinates, namely

$$\frac{d^2}{dr^2}\vec{r}(t) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta}\,,$$

and assume $r(\theta(t))$ to be the equation of an ellipse:

$$r = \frac{r_0}{1 + e\cos\theta}$$

The angular component of the acceleration is easily shown to be zero by differentiating the (constant) angular momentum, giving

$$0 = \frac{d}{dt}L = \frac{d}{dt}\left(m_* r^2 \frac{d\theta}{dt}\right) = m_* r\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\,.$$

In terms of L, the acceleration vector reads

$$\frac{d^2}{dr^2}\vec{r}(t) = \left(\frac{d^2r}{dt^2} - \frac{L^2}{m_* r^3}\right)\hat{r}\,.$$

The second derivative term is a bit messy:

$$\frac{dr}{dt} = \frac{e}{r_0} r^2 \sin\theta \frac{d\theta}{dt}$$
$$\frac{d^2r}{dt^2} = \frac{e}{r_0}\left(\sin\theta \cancel{\frac{d}{dt}\left(\frac{L}{m_*}\right)} + \left(\frac{L^2}{m_*^2 r^2}\right)\left(\frac{r_0}{e}\frac{1}{r} - \frac{1}{e}\right)\right) = \left(\frac{L^2}{m_*^2 r^2}\right)\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

The anticipated acceleration vector is thus

$$\frac{d^2}{dr^2}\vec{r}\,(t) = \left(\frac{L^2}{m_*^2 r^2}\right)\left(\cancel{\frac{1}{r}} - \frac{1}{r_0} - \cancel{\frac{1}{r}}\right)\hat{r} = -\frac{\gamma}{m_*}\frac{\hat{r}}{r^2} \; ,$$

indicating the $-1/r^2$ attraction along $\hat{r}$. Evidently, the acceleration vector of a planet in orbit swivels to point at the center of attraction.

Universal Gravitation

Having cracked the problem of planetary orbits, Newton proposed that the same mechanism be responsible for *all* celestial motions, including ground-level kinematics, unifying the notion of gravity. Between any two particles, the gravitational attraction is

$$\vec{F}_{12} = -\frac{G m_1 m_2}{r^2}\,\hat{r} \; ,$$

proportional to the product of each mass, inversely proportional to the square of the distance, and balanced by a proportionality constant G. As a special case of the two-body problem, it's readily shown that

$$\gamma = G m_1 m_2 \; .$$

Conical Orbits

As an exercise and reality check, we should be able to recover the equation for conic sections using the law of gravitation and conservation of angular momentum. Replacing r^2 as it occurs in $L = m_* r^2 d\theta/dt$, we have

$$\frac{d^2}{dt^2}\vec{r}\,(t) = \frac{d}{dt}\vec{v}\,(t) = -\frac{\gamma}{m_*}\frac{m_*}{L}\frac{d\theta}{dt}\,\hat{r} \; ,$$

which can be integrated over the time variable

$$\int \frac{d}{dt}\vec{v}\,(t)\; dt = -\frac{\gamma}{L}\int \hat{r}\, d\theta = \frac{\gamma}{L}\int \frac{d\hat{\theta}}{d\theta}\, d\theta \; ,$$

resulting in

$$\vec{v}\,(t) = \frac{\gamma}{L}\,\hat{\theta}\,(t) + \vec{V} \; ,$$

where $\vec{V}$ is an integration constant. Define the x-axis to pass through a perigee at which velocity is purely along $\hat{y}$, thus the integration constant points that way:

$$\vec{v}\,(t) = \frac{\gamma}{L}\,\hat{\theta}\,(t) + V\,\hat{y}$$

Next, write the angular momentum vector $\vec{L} = m_* \vec{r} \times \vec{v}$ in full form and cancel all of the $\hat{z}$ unit vectors:

$$L\,\hat{z} = m_* \vec{r} \times \left(\frac{\gamma}{L}\,\hat{\theta} + V\,\hat{y}\right) = m_* \frac{\gamma}{L} r\left(\hat{r} \times \hat{\theta}\right) + m_* V r\left(\hat{r} \times \hat{y}\right)$$
$$L = \frac{\gamma m_*}{L} r + m_* V r \sin\left(\frac{\pi}{2} - \theta\right) = m_* r\left(\frac{\gamma}{L} + V\cos\theta\right)$$

Indeed we find

$$r = \frac{r_0}{1 + (m_* r_0 V/L)\cos\theta}\,,$$

where $m_* r_0 V/L$ is the eccentricity of the orbit,

$$e = \frac{m_* r_0 V}{L} = \frac{VL}{\gamma}$$

classifying the overall shape.

Shell Theorem - Outside

So far, we have assumed that the 'particles' involved in central force interactions are infinitely small, with all mass concentrated at its center. Realistically though, celestial bodies are (at best) approximately spheres, so care must be taken to make sure we haven't gone astray. Newton was plagued by the same concern, which held up publication of his work for several years.

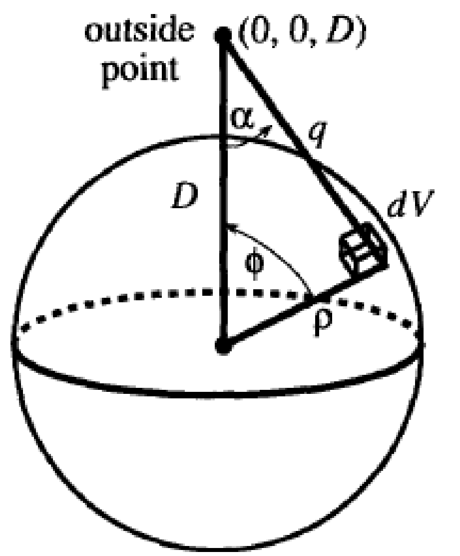

Figure 1: Particle outside of a sphere. (Image credit: *Calculus*, Gilbert Strang)

In the Figure, we have a solid sphere of radius R having uniform mass per volume $\lambda = dM/dV = M/\left(4\pi R^3/3\right)$. A test particle at point $D\,\hat{z}$ from the center 'feels' the gravitational attraction all elements dV, each being distance q from the particle and ρ from the center. For convenience, we label the angle between D and ρ as ϕ, and similarly the angle between D and q as α.

By placing the test particle of mass m on the z-axis, the net attraction in the x- and y-directions resolves to zero by symmetry. The net force is the integral

$$F = \int_{sphere} d\vec{F} \cdot \hat{z} = \int_{sphere} dF \cos\alpha = -Gm\lambda \int_{sphere} \frac{\cos\alpha}{q^2}\, dV \ .$$

Next, we write two geometric observations using the law of cosines

$$\cos\alpha = \frac{q^2 + D^2 - \rho^2}{2qD} \qquad\qquad u = q^2 = \rho^2 + D^2 - 2\rho D \cos\phi \ ,$$

and also use spherical coordinates such that

$$dV = \rho^2 \, d\rho \, d\theta \, \sin\phi \, d\phi \ .$$

The integral becomes:

$$\begin{aligned}
F &= -\frac{Gm\lambda}{2D} \int_0^{\pi} \int_0^{\pi/2} \int_0^{R} \left(\frac{1}{q} - \frac{D^2 - \rho^2}{q^3} \right) \rho^2 \, d\rho \, d\theta \, \sin\phi \, d\phi \\
F &= -\frac{Gm\lambda\pi}{2D^2} \int_{(D-\rho)^2}^{(D+\rho)^2} \int_0^{R} \left(\frac{1}{\sqrt{u}} - \frac{D^2 - \rho^2}{u^{3/2}} \right) \rho \, d\rho \, du \\
F &= -\frac{Gm\lambda\pi}{2D^2} \int_0^{R} \left(2\sqrt{u} - \frac{2\,(D^2 - \rho^2)}{\sqrt{u}} \right) \Bigg|_{(D-\rho)^2}^{(D+\rho)^2} \rho \, d\rho \\
F &= -\frac{Gm\lambda\pi}{2D^2} \int_0^{R} 8\rho^2 \, d\rho = -\frac{Gm\lambda}{D^2} \frac{4\pi R^3}{3} \\
F &= -\frac{GMm}{D^2}
\end{aligned}$$

Amazingly, the force acts as if *all* of the sphere's mass is at the center. Incidentally this is true for non-spherical objects as well, as long as the test particle is outside of the object.

Shell Theorem - Inside

The question of 'what does gravity feel like *inside* a planet' is solved by the shell theorem. If we consider a spherical shell of fixed radius R, we can pose a similar question with $D < R$, i.e., placing the test mass within the shell. The problem setup is more-or-less the same as the above, however to emphasize the hollowness of the shell we'll take an area integral over differential rings such that

$$dA = 2\pi R^2 \sin\phi \, d\phi \ .$$

The force integral becomes

$$F = -Gm\sigma \int_{shell} \frac{\cos\alpha}{q^2} \, dA \ ,$$

where the surface density is given by $\sigma = M/4\pi R^2$. Then, using the same geometry as above, we have

$$F = -\frac{GmM}{4D}\int_0^{\pi/2}\left(\frac{1}{q} - \frac{D^2-\rho^2}{q^3}\right)\sin\phi\, d\phi\,.$$

Noting that

$$2q\, dq = 2RD\sin\phi\, d\phi\,,$$

the integral becomes elementary

$$F = -\frac{GmM}{4RD^2}\int_{R-D}^{R+D}\left(1 - \frac{D^2-R^2}{q^2}\right)\, dq\,,$$

easily shown to be *zero*:

$$F_{\text{inside}} = 0$$

Indeed, a test particle anywhere within a hollow shell feels no attraction.

Problems

1. A particle has a known trajectory

$$r\,(t) = \frac{r_0}{\cos\left(\omega t\right)}\,,$$

   where ω is constant. Graph the motion and determine the velocity vector and the acceleration vector.

2. Show that Kepler's law of equal areas hold for any central force, including straight-line motion.

3. A missile traveling at constant speed is homing in on a target at the origin. Due to an error in its circuitry, it is constant misdirected by an angle $|\alpha| < \pi/2$. Show that the missile eventually hits the target, taking $1/\cos\alpha$ times as long as if it was correctly aimed.

4. If a planet were suddenly stopped in its orbit, supposed circular, show that it would fall into the sun in a time $\tilde{T}$ which is $\sqrt{2}/8$ times the period of the planet's revolution.

Solutions

1. ...

2. ...

3. ...

4.

$$E_0 = -\frac{GMm}{R} = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - \frac{GMm}{r\left(t\right)}$$

$$\frac{1}{2}\cancel{m}\left(\frac{dr}{dt}\right)^2 = GM\cancel{m}\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$\frac{dr}{dt} = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{R}\right)} = \sqrt{\frac{2GM}{R}}\sqrt{\frac{R}{r} - 1}$$

$$\int_R^0 \frac{dr}{\sqrt{R/r - 1}} = \sqrt{\frac{2GM}{R}}\int_0^{\tilde{T}} dt = \sqrt{\frac{2GM}{R}}\,\tilde{T}$$

$$r = R\cos^2\theta \qquad \to \qquad -2R\cancel{\left(\int_\pi^{\frac{\pi}{2}} \cos^2\theta\, d\theta\right)}\left(\frac{-\pi}{4}\right) = \sqrt{\frac{2GM}{R}}\tilde{T}$$

$$\tilde{T} = \frac{\pi R}{2}\sqrt{\frac{R}{2GM}} = \frac{\sqrt{2}}{8}\frac{2\pi}{\sqrt{GM}}R^{3/2} = \frac{\sqrt{2}}{8}\,T$$