Calculus

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1 Limits

In the development of elementary algebra, the star players turn out to be real numbers, operators, equations, variables, and functions. Using the standard toolkit, we never think twice about the validity of algebra, that is, until a strange situation such as division by zero arises, in which case the pursuit is usually deemed meaningless. Here we develop the notion of *mathematical limits*, which are a means of understanding situations when ordinary algebra 'blows up'.

1.1 Algebraic Analysis

Consider the function

$$y(x) = \frac{x^2 - 3x + 2}{x - 1} \,,$$

and let us inquire what the value of $y(x)$ becomes when $x = 1$. By substitution, we clearly have

$$y(1) = \frac{1^2 - 3 + 2}{1 - 1} = \frac{0}{0} \,,$$

which is indeterminate.

By pure luck however, the numerator can be factored via

$$y(x) = \frac{(x-2)(x-1)}{x-1} = \frac{(x-2)\cancel{(x-1)}}{\cancel{(x-1)}} \,,$$

and the denominator has now canceled. This in effect transforms $y(x)$ into an equivalent function $\tilde{y}(x)$ that allows $x = 1$ with no indeterminacy. Simply trusting the symbols, let us evaluate $\tilde{y}(1)$ to find

$$\tilde{y}(1) = 1 - 2 = -1 \,.$$

Of course, the modified function $\tilde{y}(x)$ is the same as the original $y(x)$. By calculating $\tilde{y}(1) = -1$, we can also argue that $y(1) = -1$, but not *directly*. In precise language, we have skirted around the division-by-zero problem by calculating the *limit* of $y(x)$ as x becomes arbitrarily close to one, but actually 'skips over' the exact value:

$$y(1) = \frac{0}{0} \qquad\qquad \lim_{x \to 1} y(x) = -1$$

1.2 Numeric Analysis

Let us take the same example function

$$y(x) = \frac{x^2 - 3x + 2}{x - 1} \,,$$

and check its numerical value near $x = 1$, leading to the table below:

Value x	**Ratio** $y(x)$
0.7	-1.3
0.8	-1.2
0.9	-1.1
0.99	-1.01
0.999	-1.001
1	?
1.001	-0.999
1.01	-0.99
1.1	-0.9
1.2	-0.8
1.3	-0.7

Starting in the neighborhood of $x \approx 1$, we see the result $y(x)$ steadily creeping toward -1 from *both* the increasing and decreasing directions. That is, starting from $x > 1$ or from $x < 1$, we inevitably land at

$$\lim_{x \to 1} y(x) = -1 ,$$

in agreement with the algebraic analysis. To denote from which direction x is 'moving' toward $x = 1$, we use the plus ($+$) and minus ($-$) scripts as shown:

$$\lim_{x \to 1^-} y(x) = -1 \qquad \lim_{x \to 1^+} y(x) = -1$$

Problem 1

Consider the strange exponential function

$$y(x) = x^x .$$

Use numeric analysis to calculate the limit

$$\lim_{x \to 0^+} y(x) .$$

Solution 1

Begin by calculating a few numerical results as shown:

$$.1^{.1} = 0.794328...$$
$$.01^{.01} = 0.9549926...$$
$$.001^{.001} = 0.99311605...$$
$$.0001^{.0001} = 0.999079390...$$

A discernible trend is evident: as x gets closer to zero, the result gets closer to 1. The limiting case of this is written as

$$\lim_{x \to 0^+} x^x = 1 ,$$

or more boldly,

$$0^0 = 1 .$$

1.3 Graphic Analysis

Returning again to the equation

$$y\left(x\right)=\frac{\left(x-2\right)\left(x-1\right)}{x-1}\ ,$$

we use plotting software to find a straight-line plot as shown, with the point $x=1$ removed from the domain.

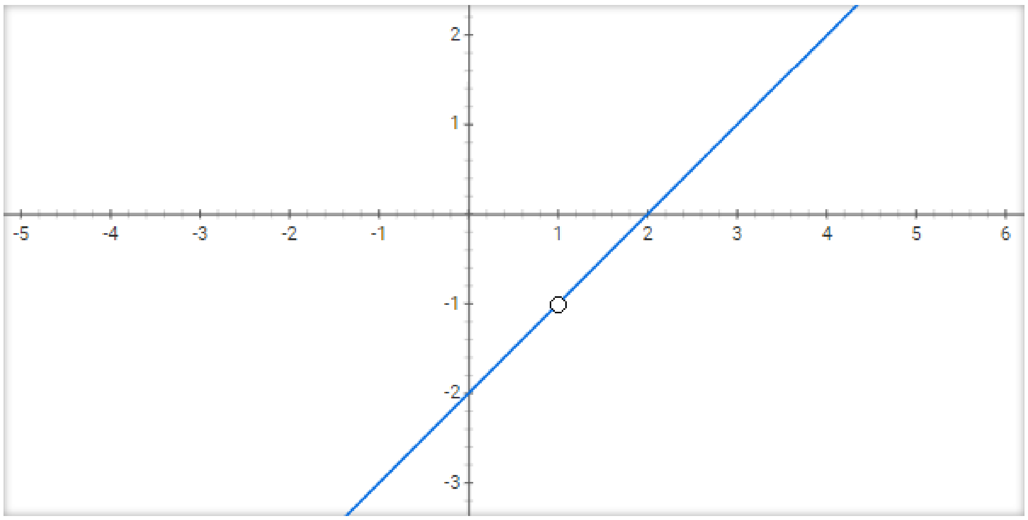

Figure 1: Plot of $y\left(x\right)=\left(x^{2}-3x+2\right)/\left(x-1\right)$ with $x=1$ removed.

In the above, it's clear that $y\left(x\right)$ is 'trying' to fill the point $\left(1,-1\right)$. Capitalizing on our successful efforts to calculate $y\left(x\rightarrow 1\right)=-1$ however, we may effectively 'fill in' the missing point as shown below:

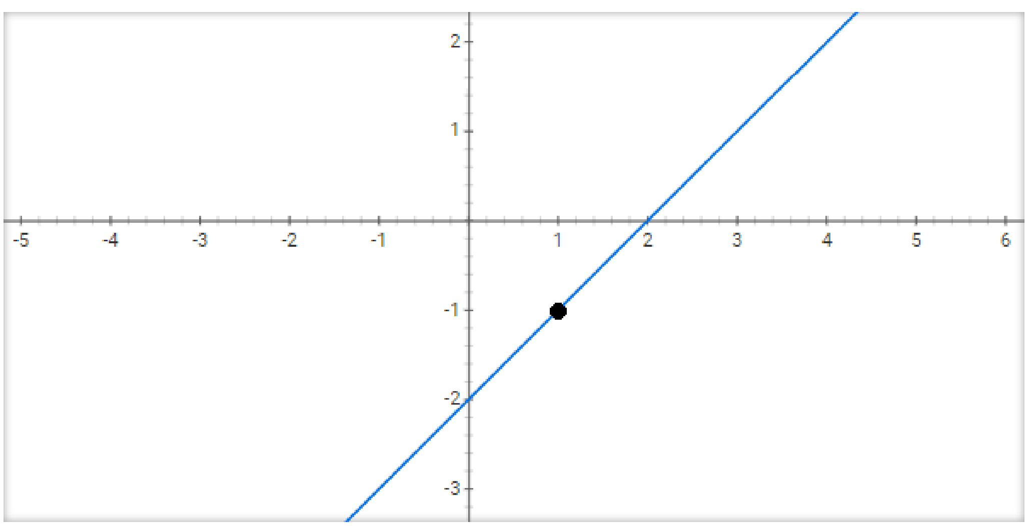

Figure 2: Plot of $y\left(x\right)=x-2$ with $x=1$ restored.

2 Continuity

A function is said to be *continuous* in a domain if each value $f(x)$ is 'touching' its neighboring values $f(x \pm \Delta x)$ for arbitrarily small Δx. Any points that correspond to a 'jump', or a 'missing point' are called *discontinuous* points in the function.

2.1 Removable Discontinuity

Consider a function $f(x)$ with a discontinuous point at x_0. Examining the value of the function to the 'left' and 'right' of x_0, we can calculate two limits

$$F_{x_0^-} = \lim_{x \to x_0^-} f(x) \qquad\qquad F_{x_0^+} = \lim_{x \to x_0^+} f(x) \ ,$$

which correspond to evaluating $f(x)$ approaching x_0 from $x < x_0$ and $x > x_0$, respectively. If it turns out that

$$F_{x_0^-} = F_{x_0^+} = F_{x_0} \ ,$$

then $f(x)$ is essentially the same on either side of x_0, thus it makes sense to assign

$$f(x_0) = F_{x_0} \ .$$

In such a case, x_0 is called a *removable discontinuity.*

2.2 Jump Discontinuity

Piece-wise functions naturally exhibit at least one *jump discontinuity*, where the function suddenly 'jumps' from one value to another at x_0. For instance, the function

$$f(x) = \begin{cases} x+1 & x > 0 \\ -x-1 & x < 0 \end{cases}$$

has two different limits

$$F_{0^-} = \lim_{x \to x_0^-} f(x) = 1 \qquad\qquad F_{0^+} = \lim_{x \to x_0^+} f(x) = -1 \ .$$

In the general case, when there is no agreement between $F_{x_0^-}$ and $F_{x_0^+}$, there is nothing concrete to say about $f(x_0)$.

2.3 Infinite Discontinuity

Many interesting functions tend toward $\pm\infty$ somewhere within their respective domain, with common players being $1/x^2$ at $x_0 = 0$, and $\tan(x)$ at $x_0 = \pm\pi/2$, and so on. In such cases, neighboring points $f(x_0 + \Delta x)$ resolve to near-infinite values, whether they be positive or negative. This only means that $f(x_0)$ is either infinite or indeterminate, thus infinite discontinuities cannot be removed.

2.4 Endpoint Discontinuity

Certain functions stop making ordinary sense outside of their respective domains. For instance, the square root function

$$f(x) = \sqrt{x}$$

is perfectly well-behaved for all $x > 0$. Of course for $x < 0$, the square root produces imaginary numbers. In terms of limits, this means

$$F_{0^-} = \lim_{x \to 0^-} \sqrt{x} = \text{imaginary} \qquad\qquad F_{0^+} = \lim_{x \to 0^+} \sqrt{x} = \text{real}\,.$$

With $f(x)$ being imaginary (troublesome) on the left, and real (well-behaved) on the right, we therefore call $x = 0$ an *endpoint discontinuity*. What then, do we make of the exact value of $f(0) = \sqrt{0}$? The $\sqrt{x}$ function tends to zero for small values of x, thus we may safely assume the limit case

$$\sqrt{0} = 0\,.$$

Problem 2

Classify all discontinuities of the function

$$y(x) = \frac{x-1}{x^2-1}\,.$$

Solution 2

Begin by finding the discontinuous points, most easily done by factoring the denominator:

$$y(x) = \frac{(x-1)}{(x+1)(x-1)}$$

The function risks discontinuity at two points, namely $x = -1$ and $x = 1$. Observe however that the numerator cancels one of them, meaning $x = 1$ is a removable discontinuity. The function simplifies to

$$y(x) = \frac{1}{x+1}\,,$$

which contains an infinite discontinuity at $x = -1$.