Algebra

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Contents

1 Numbers and Operations

1.1 Real Numbers

Numbers are symbols used for counting and measuring: 0, 2/3, 1, −1.5, 2, etc. These have a formal name called *real numbers*, which is the set of all negative and positive numbers (including zero) spanning both directions on the number line. Real numbers include all fractions and decimal numbers:

$$-\infty < \text{Real Numbers} < \infty$$

Integers are a synonym for whole numbers, which are real numbers without fractions or decimals. Just like the real numbers, the integers include zero and span the whole number line:

$$-\infty < \ldots, -3, -2, -1, 0, 1, 2, 3, \cdots < \infty$$

1.2 Operators and Precedence

Numbers may be combined or manipulated using mathematical *operators*, which go between numbers, and *bracketing symbols*, which go around numbers:

Bracketing symbols:	( ) [ ] { }
Operators:	ˆ × / + −

When mathematical statements contain more than one operator, each operation is evaluated in a specific order, called the *order of operations*. The order is contained in the made-up word 'PEMDAS', which stands for Parentheses, Exponents, Multiplication, Division, Addition, Subtraction. This can be remembered by the phrase **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally:

Parentheses	( )	go around numbers.
Exponent	ˆ	results in a product.
Multiplication	×	results in a product.
Division	/	results in a ratio or quotient.
Addition	+	results in a sum.
Subtraction	−	results in a difference.

1.3 Expressions

A mathematical *expression* is any valid combination of numbers or variables joined by bracketing symbols and operators. Expressions can be reduced to equivalent expressions by executing operations according to their precedence (PEMDAS).

Example 1

Evaluate the expression: $4 + 5 \times 9$

Look for parentheses.	(None.)
Look for exponents.	(None.)
Recognize 5×9 as the next valid operation.	$5 \times 9 = 45$
Rewrite the problem (optional).	$4 + 45$
Recognize $4 + 45$ as the next valid operation.	$4 + 45 = 49$
Write the result and check.	$4 + 5 \times 9 = 49$

Example 2

Evaluate the expression: $(4 + 5) \times 9$

Look for parentheses and evaluate the contents.	$(4 + 5) = 9$
Look for exponents.	(None.)
Rewrite the problem (optional).	$(9) \times 9$
Recognize 9×9 as the next valid operation.	$9 \times 9 = 81$
Write the result and check.	$(4 + 5) \times 9 = 81$

Binomials and Trinomials and Polynomials (oh my!)

The simplest kind of mathematical expression is a *binomial*, which contains two numbers, or *terms*, separated by a plus or minus sign. For instance,

$$2 + 4 \qquad\qquad 3 - 8$$

are binomials, whereas each of

$$3 - 4 + 1 \qquad\qquad 7 + 11 - 6$$

are classified as *trinomial* due to having to three terms and top operators. In general, expressions containing multiple terms are called *polynomials*.

Role of Zero

You already know what 'zero' does in an intuitive sense, so let us jot down its role in complete statements. In the following, the letter A stands for 'Any' number.

- Adding or subtracting zero to any quantity leaves the quantity unchanged.

$$A \pm 0 = A$$

- Multiplying any quantity by zero results in zero.

$$A \times 0 = 0$$

- Division by zero produces no useful information.

$$\frac{A}{0} = \text{Undefined}$$

Role of One

Meanwhile, a shorter list can be written about the number one:

- Multiplying any quantity by one leaves the quantity unchanged.

$$A \times 1 = A$$

- Dividing any quantity by one leaves the quantity unchanged.

$$A \div 1 = A$$

1.4 Prime Numbers

Prime numbers are integers that cannot be divided into smaller integers (besides 1). Apart from 2, no even numbers are prime. There is otherwise no simple pattern to the prime numbers, indicated by bold script in the following table. (Prime numbers don't seem to follow a discernible pattern.)

$$\begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 & 9 & 10 \\ \mathbf{11} & 12 & \mathbf{13} & 14 & 15 & 16 & \mathbf{17} & 18 & \mathbf{19} & 20 \\ 21 & 22 & \mathbf{23} & 24 & 25 & 26 & 27 & 28 & \mathbf{29} & 30 \\ \mathbf{31} & 32 & 33 & 34 & 35 & 36 & \mathbf{37} & 38 & 39 & 40 \\ \mathbf{41} & 42 & \mathbf{43} & 44 & 45 & 46 & \mathbf{47} & 48 & 49 & 50 \end{bmatrix}$$

Prime Decomposition

The *prime decomposition* of a number is defined as a list of prime numbers which when multiplied together, produce the original number. Reading this statement backwards, it follows that any integer can be broken apart or 'decomposed' into prime numbers. To find the prime decomposition of an odd number, try dividing by 2. The result can be one of three things:

- If the result is a prime number, you're done.
- If the result comes out to an integer (no fraction or decimal), then 2 is one of the factors. Proceed by dividing the new result by 2 and repeat.
- If the result was not a round integer, discard the 2 and try dividing by the next prime number, 3 and repeat for 5, 7, and so on.

<u>Example 3</u>

Decompose the number 12 into prime factors.

Try dividing 12 by 2:	$12/2 = 6$
Is the result an integer?	Yes, so keep the **2**.
Is the result a prime number?	No, so keep going using 6.
Try dividing 6 by 2:	$6/2 = 3$
Is the result an integer?	Yes, so keep the **2**.
Is the result a prime number?	Yes, so stop at **3**.
Write the result and check:	$12 = 2 \cdot 2 \cdot 3$

Example 4

Decompose the number 231 into prime factors.

Try dividing 231 by 2:	$231/2 = 115.5$ (not an integer)
Try dividing 231 by 3:	$231/3 = 77$
Is the result an integer?	Yes, so keep the **3**.
Is the result a prime number?	No, so keep going using 77.
Try dividing 77 by 2:	$77/2 = 38.5$ (not an integer)
Try dividing 77 by 3:	$77/3 = 25.667$ (not an integer)
Try dividing 77 by 5:	$77/5 = 15.4$ (not an integer)
Try dividing 77 by 7:	$77/7 = 11$
Is the result an integer?	Yes, so keep the **7**.
Is the result a prime number?	Yes, so stop at **11**.
Write the result and check:	$231 = 3 \cdot 7 \cdot 11$

Tree Method

A pictorial technique for prime decomposition is the so-called *tree method*, which represents the factors of a number as 'branches'. The final extremity of each branch is the set of prime factors. See the Figure below.

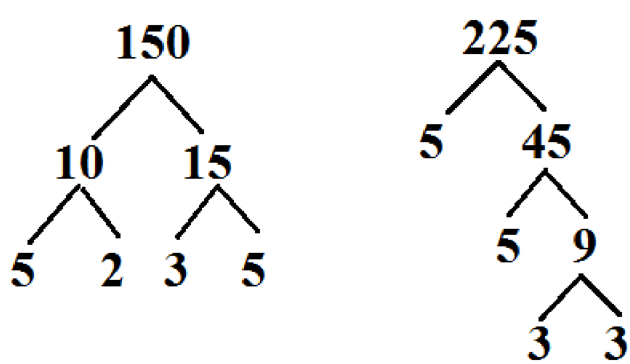


Figure 1: Tree method demonstrating $150 = 5 \times 2 \times 3 \times 5$ and $225 = 5 \times 5 \times 3 \times 3$.

1.5 Properties of Multiplication

Commutative Property

The *commutative property* contains the idea that two numbers may be multiplied in either order without changing the result. For example:

$$2 \times 3 = 6 \qquad\qquad 3 \times 2 = 6$$

Associative Property

The *associative property* tells us that the grouping of factors in a multiplication problem does not affect the result. For example:

$$2 \times 3 \times 4 = 24 \qquad 2 \times (3 \times 4) = 24 \qquad (2 \times 3) \times 4 = 24$$

Distributive Property

The *distributive property* dictates how expressions containing sums and differences in parentheses are multiplied. The idea of 'distribution' means that whatever is outside the parentheses is multiplied into *each* term enclosed inside the parentheses. For example we can multiply the number 4 into the binomial $2 - 3$ as:

$$4 \times (2 - 3) = 4 \times 2 - 4 \times 3$$

Supposing the number 4 was instead itself a binomial, such as $4 - 1$, the example becomes:

$$\begin{aligned}(4-1) \times (2-3) &= (4-1) \times 2 - (4-1) \times 3 \\ &= 4 \times 2 - 1 \times 2 - 4 \times 3 + 1 \times 3\end{aligned}$$

FOIL Method for Binomials

A trick for remembering the steps to multiply two binomials is contained in the word 'FOIL', where each respective letter stands for **F**irst, **O**uter, **I**nner, **L**ast. For instance, consider the product

$$(4+2)(5-3) \; .$$

To begin we multiply the **F**irst respective terms to get

$$4 \times 5 = 20 \, .$$

Next, multiply the **O**uter terms, resulting in

$$4 \times -3 = -12 \, .$$

Next we multiply the **I**nner terms, namely

$$2 \times 5 = 10 \, ,$$

and finally, the **L**ast terms multiply to give

$$2 \times -3 = -6 \, .$$

The final answer is the sum of the four respective products. In our example, we have

$$(4+2)(5-3) = 20 - 12 + 10 - 6 = 12 \, .$$

Of course, we this was visible from the outset, as

$$(4+2)(5-3) = (6) \times (2) = 12 \, .$$

The FOIL method works on *all* binomials, but *only* works on binomials.

Greatest Common Factor

Recall form our study of prime decomposition that any whole integer can be decomposed into the product of smaller factors, all the way down to prime numbers. The *greatest common factor* (or GCF) is the largest number that can multiply into another set of numbers. More generally, a *common factor* is any number that can multiply into two other numbers.

Example 5

Find all common factors of 12 and 30. Identify the greatest common factor.

List all factors of 12.	1, 2, 3, 4, 6, 12
List all factors of 30.	1, 2, 3, 5, 6, 10, 15, 30
Identify the common (shared) factors.	1, 2, 3, 6
Identify the greatest common factor.	6

Example 6

Find all common factors of 15, 30 and 105. Identify the greatest common factor.

List all factors of 15.	1, 3, 5, 15
List all factors of 30.	1, 2, 3, 5, 6, 10, 15, 30
List all factors of 105.	1, 3, 5, 7, 15, 21, 35, 105
Identify the common (shared) factors.	1, 3, 5, 15
Identify the greatest common factor.	15

Example 7

Use prime decomposition to find the GCF of 24 and 108.

Write prime decomposition of 24.	$24 = 2 \times 2 \times 2 \times 3$
Write prime decomposition of 108.	$108 = 2 \times 2 \times 3 \times 3 \times 3$
Identify the greatest common factor.	$2 \times 2 \times 3 = 12$

Example 8

A boy has 160 red marbles and 144 blue marbles. He wants to split the marbles into identical groups (not necessarily the same number of each color in a group). What is the greatest number of groups he can make?

This is solved by finding the GCF of 160 and 144. Working these out, find

$$160 = 2^5 \times 5 \qquad\qquad 144 = 2^4 \times 3^2\ ,$$

indicating the GCF is $2^4 = 16$, meaning 16 groups of marbles can be made. Each group has $160/16 = 10$ red marbles, and $144/16 = 9$ blue marbles.

Least Common Multiple

The *least common multiple* (or LCM) is the product of prime factors of two or more numbers, with each prime factor taken to the highest power in which it occurs. That is, the LCM is the *lowest* number that can be evenly divided by two or more numbers.

Example 9

Find the least common multiple of 84 and 147.

Write prime decomposition of 84.	$84 = 2^2 \times 3 \times 7$
Write prime decomposition of 147.	$147 = 3 \times 7^2$
List each factor to highest power.	2^2, 3^1, 7^2
The LCM is the product of each factor.	$LCM = 2^2 \times 3 \times 7^2 = 588$

Example 10

What is the lowest number that can be evenly divided by 3, 9, and 21?

Write prime decomposition of 3.	$3 = 3^1$
Write prime decomposition of 9.	$9 = 3^2$
Write prime decomposition of 21.	$21 = 3^1 \times 7^1$
List each factor to highest power.	3^2, 7^1
The LCM is the product of each factor.	$LCM = 3^2 \times 7 = 63$

Example 11

Alice works in the orchard picking peaches, and fits 8 peaches per bag. Bob has the same job, but fits 9 peaches per bag. At the end of the day, they have picked the same number of peaches. What is the smallest number of peaches they each could have picked?

This is solved by finding the LCM of 8 and 9. Note that $8 = 2^3$, and $9 = 3^2$, telling us the LCM is $2^3 \times 2^2 = 72$.

Example 12

Find the greatest common factor (GCF) and lowest common multiple (LCM) of the two numbers:

2940 3150

Decompose each number into primes:

$$2940 = 2 \times 2 \times 3 \times 5 \times 7 \times 7 = 2^2 \times 3 \times 5 \times 7^2$$
$$3150 = 2 \times 3 \times 3 \times 5 \times 5 \times 7 = 2 \times 3^2 \times 5^2 \times 7$$

The GCF is the greatest number that multiplies into 2940 and 3150. This is evidently

$$GCF = 2 \times 3 \times 5 \times 7 = 210\,.$$

Meanwhile, the LCM is the product of each above-listed prime, raised to the highest-occurring power, ignoring the lower-power occurrence. Therefore,

$$LCM = 2^2 \times 3^2 \times 5^2 \times 7^2 = 210^2\,.$$

1.6 Properties of Fractions

Any real number that is not an integer is some kind of fraction or decimal. These are numbers such as 1/3, 9.5, −0.6666666, and so on. The familiar symbol for division is the line with two dots (÷). This is equivalent to a forward-slash (/), and also equivalent to the numerator-over-denominator notation (top number over the bottom number). That is, the following are equivalent:

$$2 \div 3 \qquad\qquad 2/3 \qquad\qquad \frac{2}{3}$$

Intermediate and advanced math courses almost completely avoid the (÷) symbol, and instead represent fractions with the slash or horizontal bar. (Can't believe it? A standard computer keyboard has no 'divide-by' button, but it has two slashes!) In the same spirit, shall phase out the elementary symbol (but don't forget it of course).

Multiplying Fractions

When fractions are multiplied, first multiply all terms in the numerator, and then separately multiply all terms in the denominator. The resulting fraction is the ratio of the respective products. For example:

$$\frac{2}{3} \times \frac{5}{4} = \frac{2 \times 5}{3 \times 4} = \frac{5}{6} \qquad\qquad \frac{2}{3} \times \frac{5}{4} \times \frac{7}{9} = \frac{2 \times 5 \times 7}{3 \times 4 \times 9} = \frac{35}{54}$$

Notion of Reciprocal

The *reciprocal* or *inverse* of a fraction is the result of exchanging the numerator and the denominator. For instance, the reciprocal of 2/7 is 7/2. For another example, the following two fractions are reciprocal of one another:

$$\frac{34}{256} \qquad\qquad \frac{256}{34}$$

In the most general sense, the reciprocal of any value is one divided by that value:

$$\text{reciprocal} = \frac{1}{\text{fraction}}$$

Dividing Fractions

With a word like 'reciprocal' under our belt, we think about fractions in a more refined fashion. Consider the equivalent fractions

$$2/3 \qquad\qquad \frac{2}{3}$$

that plainly read as 'two over three' or 'two divided by three'. An equivalent interpretation of this fraction would read 'two times one third' because, division by a number is equivalent to multiplying by that number's reciprocal. For our example, this means:

$$\frac{2}{3} = 2 \times \frac{1}{3} \qquad\qquad 2/3 = 2 \times 1/3$$

Example 13

Convert to a single fraction and simplify

$$(3/4) \div (2/5)\ .$$

First, identify the reciprocal of the fraction being divided, namely

$$\frac{5}{2}\ .$$

Next, rewrite the division problem as multiplication by the reciprocal and simplify as necessary:

$$(3/4) \div (2/5) = \frac{3}{4} \times \frac{5}{2} = \frac{15}{8}$$

Example 14

Convert to a single fraction and simplify

$$\frac{3/5}{7/8}\ .$$

First, identify the reciprocal of the fraction being divided, namely

$$\frac{8}{7}\ .$$

Next, rewrite the division problem as multiplication by the reciprocal and simplify as necessary:

$$\frac{(3/5)}{(7/8)} = \frac{3}{5} \times \frac{8}{7} = \frac{24}{35}$$

Change of Denominator

Fractions can be changed to have a different numerator and denominator, but only if the numerical value of the fraction remains the same. For example, we know that

$$\frac{1}{2} \qquad\qquad \frac{3}{6}$$

are *equivalent*, and evaluate to 0.5, yet the actual numbers used in each fraction are different.

Conversion between equivalent fractions is achieved by multiplying the fraction by a carefully-chosen factor of one. For example, we convert 1/2 to 3/6 by

$$\frac{1}{2} = \frac{1}{2} \times (1) = \frac{1}{2} \times \left(\frac{3}{3}\right) = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}\ .$$

Choosing a different factor of one, we can convert to any fraction that is ultimately equivalent to the original ratio. For instance:

$$\frac{1}{2} = \frac{1}{2} \times (1) = \frac{1}{2} \times \left(\frac{13}{13}\right) = \frac{1 \times 13}{2 \times 13} = \frac{13}{26}$$

Example 15

Consider the ratio 9/3. Change the denominator to 17 without changing the numerical value of the fraction.

$$\frac{9}{3} = \frac{9/3}{1} \times (1) = \frac{9/3}{1} \times \left(\frac{17}{17}\right) = \frac{9/3 \times 17}{17} = \frac{51}{17}$$

Example 16

Consider the two fractions 2/3 and 3/4. Rewrite each fraction such that the denominators are the same.

The LCM of 3 and 4 is 12, thus the target denominator is 12. Thus each fraction becomes:

$$\frac{2}{3} = \frac{2}{3} \times \left(\frac{4}{4}\right) = \frac{8}{12} \qquad\qquad \frac{3}{4} = \frac{3}{4} \times \left(\frac{3}{3}\right) = \frac{9}{12}$$

Adding and Subtracting Fractions

There is one single rule for adding and subtracting fractions, and that is: only fractions having the *same* denominator can be added or subtracted. For instance, consider the true statement:

$$\frac{3}{4} - \frac{1}{4} = \frac{3-1}{4} = \frac{2}{4}$$

The reason the above statement executes without error is the denominator of each fraction being added (subtracted in this case) is the same. On the left side, the two terms 3/4 and 1/4 share a denominator (namely 4), thus the two terms combine easily, which begs the question: what if the denominators not the same? The answer is to convert one (or both) of the fractions such that the denominators match.

Example 17

Evaluate the sum of 2/3 and 3/4.

These fractions can only be summed when the denominators are equal. Borrowing from the Example above, we have

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}\,.$$

Example 18

Evaluate the expression: $1/2 + 1/3 - 1/7$

The LCM of each denominator, namely, 2, 3, and 7 is 42, hence

$$\frac{1}{2} + \frac{1}{3} - \frac{1}{7} = \frac{21}{42} + \frac{14}{3} - \frac{6}{42} = \frac{29}{42}\,.$$

1.7 Properties of Exponents

Exponent Notation

When a number is multiplied by itself more than twice, or many times, we don't want to be burdened by writing things like $6 \times 6 \times 6 \times 6$. This is remedied by exponent notation, demonstrated as follows:

$$
\begin{aligned}
4 &= 4^1 \\
3 \times 3 &= 3^2 \\
5 \times 5 \times 5 &= 5^3 \\
2 \times 2 \times 9 \times 9 &= 2^2 \times 9^2
\end{aligned}
$$

Exponents obey certain conventions that we shall establish here.

- The *base* number is the 'lower' number to which the exponent applies.
- The *exponent* is written smaller and to the upper-right of the base number.
- When the exponent is 2, the result is the *square* of the base number.
- When the exponent is 3, the result is the *cube* of the base number.

Negative Base Numbers

Negative numbers with exponents must be treated carefully. For instance, if asked to square the number -3, one may write

$$(-3)^2 = -3 \times -3 = (-1) \times (-1) \times 3^2 = 9 \,,$$

which has the same result as (positive) $3^2 = 9$. Indeed, raising to any base number *even* power results in a positive number. On the other hand, if a negative base number is raised to an *odd* number, factors of -1 occur an odd number of times, as in

$$(-2)^3 = (-1) \times (-1) \times (-1) \times 2^3 = -8 \,.$$

Beware the placement of parentheses when handling exponents. For instance the number -2^4 translates to $-2\times2\times2\times2 = -16$. However, if we were handed $(-2)^4$, the full statement would be $(-2) \times (-2) \times (-2) \times (-2) = 16$. (The two answers differ by a sign!)

Multiplying Numbers with Exponents

For two numbers having equal base and unequal exponents, their product is the base number raised to the sum of the exponents. For example, the product $3^2 \times 3^4$ can be immediately translated to 3^6:

$$(3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^{2+4} = 3^6$$

Negative Exponents

A number raised to a negative exponent is equivalent to the reciprocal of that number raised to the positive exponent. For example, 3^{-4} can be immediately translated to $(1/3)^4$. This is consistent with the notion of adding (or subtracting) exponents when combining numbers of similar base. For instance,

$$\begin{aligned} 3^2 \times 3^4 &= 3^6 \\ \cancel{\left(\frac{1}{3^4}\right)} \times 3^2 \times \cancel{3^4} &= \left(\frac{1}{3^4}\right) \times 3^6 \\ 3^2 &= 3^{6-4} \end{aligned}$$

Compounded Exponents

A number that is raised to an exponent, and then raised to an exponent *again*, as in

$$\left(2^3\right)^2 = 8^2 = 64$$

is equivalent to *multiplying* the two exponents:

$$\left(2^3\right)^2 = 2^{3\times 2} = 2^6 = 64$$

Zero Exponent

A number that is raised to an exponent of *zero*, results in precisely *one*. For instance, the product of 3^2 and 3^{-2} resolves to 1:

$$\frac{3^2}{3^2} = 3^2 \times 3^{-2} = 3^0 = 1$$

Non-Integer Exponents

Numbers raised to non-integer exponents are generally called *radicals*. Valid radical numbers may look like:

$$3^{.5} = \sqrt{3} \qquad 4^{3/2} = \sqrt[3]{4} \qquad 7^{-1.5} = 7^{-3/2} = \frac{1}{7^{3/2}} = \frac{1}{\sqrt[3]{7}}$$

The base number is also denoted the *radicand*, and the exponent is called the *degree* of the radical.

When the exponent is precisely 1/2, the result of the operation is the *square root* of the base number. Similarly, if the exponent is precisely 1/3, the result of the operation is the *cube root* of the base number, and so on. Since the square root is so common, we may as well generate the following reference table using a calculator:

x	$\sqrt{x}$	x	$\sqrt{x}$
1	1	26	5.0990195135
2	1.4142135620	27	5.1961524227
3	1.7320508075	28	5.2915026221
4	2	29	5.3851648071
5	2.2360679774	30	5.4772255750
6	2.4494897427	31	5.5677643628
7	2.6457513110	32	5.6568542494
8	2.8284271247	33	5.7445626465
9	3	34	5.8309518948
10	3.1622776601	35	5.9160797830
11	3.3166247903	36	6
12	3.4641016151	37	6.0827625302
13	3.6055512754	38	6.1644140029
14	3.7416573867	39	6.2449979983
15	3.8729833462	40	6.3245553203
16	4	41	6.4031242374
17	4.1231056256	42	6.4807406984
18	4.2426406871	43	6.5574385243
19	4.3588989435	44	6.6332495807
20	4.4721359549	45	6.7082039324
21	4.5825756949	46	6.7823299831
22	4.6904157598	47	6.8556546004
23	4.7958315233	48	6.9282032302
24	4.8989794855	49	7
25	5	50	7.0710678118

1.8 Problems

Problem 1

The set of real numbers contains

(A) only positive numbers from zero to infinity.

(B) whole numbers but not fractions.

(C) all integers and decimals.

Problem 2

The only even prime number is:

(A) 0

(B) 2

(C) 4

(D) None, there are no even primes.

Problem 3

The prime factors of 15 are:

(A) $5+5+5$

(B) $5, 10$

(C) $3, 5$

Problem 4

Use any method to write the prime factors of:

300 160 99

Problem 5

Evaluate the product:

$$2 \times \left(1+3^2\right)$$

Problem 6

Evaluate the ratio:

$$\frac{-6+4^2}{(5-3)}$$

Problem 7

Evaluate the expression:

$$-2^3 \times \left((5+5)-3^2\right)+9$$

Problem 8

Evaluate the expression:

$$-4+4-4\times 4+4^{(1+1)}$$

Problem 9

The rule that guarantees $3 \times 4 = 4 \times 3$ is the

(A) distributive property.

(B) associative property.

(C) prime factorization.

Problem 10

The rule that guarantees $2 \times (3 \times 4) = (2 \times 3) \times 4$ is the

(A) distributive property.

(B) associative property.

(C) prime factorization.

Problem 11

Use the distributive property to discern which statement is *false*:

(A) $(7-3)\times 2 = 7\times 2 - 3\times 2$

(B) $-5 = -5\times(2-3)$

(C) $2+(3\times 4) = 2+3\times 2+4$

(D) $5\times(2+3) = 25$

Problem 12

Use the FOIL method to evaluate:

- $(7-4)(6+3)$
- $(4+5)(4-5)$
- $(3^2-7)(3-2)$
- $(-3+7)(-2-5)$

Problem 13

List all factors of 150 and 225. Identify the greatest common factor (GCF).

Problem 14

A boy has 160 red marbles and 144 blue marbles. He wants to split the marbles into identical bags.

(A) If the boy wants to maximize the number of bags, he needs to calculate

  (a) the GCF.

  (b) the LCM.

  (c) the common denominator.

(B) Calculate the GCF of 160 (red marbles) and 144 (blue marbles).

(C) The GCF is interpreted as the

  (a) number of red marbles per bag.

  (b) number of blue marbles per bag.

  (c) total number of bags.

(D) Calculate the number of marbles in each bag.

Problem 15

Cynthia is preparing Halloween treats for her classmates. She has 72 orange candies and 24 red candies in total, and wants to divide them into the greatest number of bags. How many bags can she prepare, and how much of each type of candy should be in each bag?

Problem 16

Alice has 9 forks and 6 spoons. She intends to lay the utensils out in groups around the dinner table with none left over. How many groups can she make, and how many of each utensil per group?

Problem 17

- Find the least common multiple (LCM) of 6 and 8.
- Find the LCM 6 and 15.
- Find the LCM 4, 6 and 8.
- Prove that the LCM of 14, 16, and 18 equals 1008.

Problem 18

Two lazy basketball players start dribbling their own basketball at the same time. If one player dribbles every 7 seconds, and the other player dribbles every 8 seconds, how long until both basketballs hit the floor at the same time?

Problem 19

Betty is thinking of a number divisible by 17 and 8. What is the smallest number that Betty could be thinking of?

Problem 20

Convert to a single fraction and simplify:

$$\frac{4}{9} \div \frac{3}{7}$$

Problem 21

Convert to a single fraction and simplify:

$$\frac{12}{3} \times \frac{3}{2} \div \frac{7}{4}$$

Problem 22

Convert to a single fraction and simplify:

$$\frac{3/4}{5/2} \div \frac{2}{7}$$

Problem 23

Convert to a single fraction and simplify:

$$\frac{1}{7} \div \frac{2/3}{4/9}$$

Problem 24

Consider the fraction 4/5. Change the denominator to 20 without changing the numerical value of the fraction.

Problem 25

Consider the fraction 9/15. Change the denominator to 3 without changing the numerical value of the fraction.

Problem 26

Consider the fraction 2/3. Change the denominator to 5 without changing the numerical value of the fraction.

Problem 27

Consider the fraction 9/15. Change the denominator to 13 without changing the numerical value of the fraction.

Problem 28

Evaluate the sum:

$$\frac{4}{5} + \frac{7}{9}$$

Problem 29

Evaluate the difference:

$$\frac{1}{4} - \frac{2}{3}$$

Problem 30

Simplify:

$$\frac{4}{9} + \frac{1}{3} - \frac{4}{18}$$

Problem 31

Simplify:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7}$$

Problem 32

Simplify:

$$4^2 \times 4^3 \times 4^{-2}$$

Problem 33

Simplify:

$$\frac{2^3}{2^6} \times 3^{-2} \times 3^4$$

Problem 34

Simplify:

$$\frac{1}{8} \times \frac{\left(4 \times 2^2\right)^2}{4^2} - \frac{1/4}{2^{-3}}$$

Problem 35

Simplify:

$$(1 \times 2 \times 3)^{(1+2+3)} \times 3^{-6}$$

Problem 36

Simplify:

$$(144)^{1/2} \times (81)^{1/2}$$

Problem 37

Simplify:

$$(16 \times 25)^{1/2}$$

Problem 38

Simplify:

$$(8)^{1/3} \times (49)^{1/2}$$

Problem 39

Insert any combination of parentheses and operators to make the following statement true:

$$1\ 1\ 1\ 1 = 5$$

Solution 39

$$(1 + 1 + 1)! - 1 = 5$$

Problem 40

Insert any combination of parentheses and operators to make the following statement true:

$$2\ 0\ 1\ 6 = 1$$

Solution 40

$$\frac{(2 + 0 + 1)!}{6} = 1$$

Problem 41

Insert any combination of operators to make the following statement true:

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9 = 100$$

Solution 41

$$1 \times 2 - 3 + 4 - 5 + 6 + 7 + 89 = 100$$
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100$$

2 Variables

A *variable* is any symbol used to represent a number (or function, or anything). Common symbols used are familiar letters x, y, z, a, b, c - and some unfamiliar (Greek) letters α, β, γ, etc. Variables behave exactly as numbers do: any place where a number is needed, a variable will work just as well.

2.1 Assignment

To *set a variable* is to associate a number with a symbol. For example, the statement $x = 3$ will 'tie' to the value of 3 to the symbol x. Any time x occurs, it really means 3. The value of a variable is the number represented by the symbol. That is, we say 3 is the 'value' of x.

Variable names such as x, y, z, etc. are heavily recycled. That is, the same letters are prone to show up in different problems. Needless to mention, variables in one problem are not the same in a different problem. (They wouldn't be variables otherwise, would they?)

2.2 Coefficients

When a variable is multiplied by a number or another variable, the multiplication symbol is usually ignored. This means $4 \times x$ or $4 \cdot x$ should be written $4x$. When a variable appears alone, there is always an invisible 1 next to it. That is, x is equivalent to $1x$ or $x/1$. The number multiplying the variable is called a *coefficient.* In the following three example terms, anything not precisely x is a coefficient, namely 3, $-1/3$, and a, respectively:

$$3x \qquad -x/3 \qquad a \cdot x$$

2.3 Combining Like Terms

When an expression contains two or more instances of the same variable raised to the same exponent, the coefficients may be added (or subtracted) in a process called *combining like terms.* For example, consider the expression:

$$2x - 5x + 7x^2$$

Notice that $2x$ and $-5x$ each contain a single instance of x. It follows that these terms may be added together to give $-3x$. The $7x^2$ term can't be combined with anything, because there are no other x^2 terms in play. The simplified expression is then:

$$-3x + 7x^2$$

2.4 Adding Polynomials

The addition and subtraction of binomials, trinomials, and polynomials is a straightforward application of combining like terms.

Example 1

Add the two trinomials:

$$\left(2x^2 + 2y^2 - 3\right) + \left(y^2 - 9x^2 - 2\right)$$

Step 1: Remove unnecessary parentheses:

$$2x^2 + 2y^2 - 3 + y^2 - 9x^2 - 2$$

Step 2: Rearrange to group like terms:

$$2x^2 - 9x^2 + 2y^2 + y^2 - 3 - 2$$

Step 3: Combine like terms:

$$-7x^2 + 4y^2 - 5$$

Example 2

Add the two polynomials:

$$\left(2x^2 + 2y^2 - 3\right) - \left(2y^2 - 4x^2\right)$$

Step 1: Distribute the minus sign into the second term:

$$\left(2x^2 + 2y^2 - 3\right) + \left(-2y^2 + 4x^2\right)$$

Step 2: Remove unnecessary parentheses:

$$2x^2 + 2y^2 - 3 - 2y^2 + 4x^2$$

Step 3: Rearrange to group like terms:

$$2x^2 + 4x^2 + 2y^2 - 2y^2 - 3$$

Step 4: Combine like terms:

$$6x^2 - 3$$

2.5 Multiplying Polynomials

Multiplying two polynomials is an application of the distributive property, which entails multiplying *every* term in the first polynomial into *every* term in the second polynomial, and adding all results. For instance, in the product

$$(y - 4)\left(y^2 - 3x - 2\right) ,$$

we may proceed by distributing y, and then -4 separately into the second polynomial

$$\begin{aligned} (y - 4)\left(y^2 - 3x - 2\right) &= y\left(y^2 - 3x - 2\right) - 4\left(y^2 - 3x - 2\right) \\ &= y^3 - 3xy - 2y - 4y^2 + 12x + 8 , \end{aligned}$$

and we're done. Alternatively, we may instead distribute the terms of the second polynomial into the first to get the same result:

$$\begin{aligned} \left(y^2 - 3x - 2\right)(y - 4) &= y^2(y - 4) - 3x(y - 4) - 2(y - 4) \\ &= y^3 - 4y^2 - 3xy + 12x - 2y + 8 \end{aligned}$$

2.6 Pascal's Triangle

An interesting pattern can be discovered by expanding each power of $(a+b)^n$, where $n = 0, 1, 2, 3, \ldots$. Starting with $n = 0$ and going upward, we find:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$
$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$
$$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

Plucking out only the coefficients on the right side, we can arrange them in *Pascal's triangle*:

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

If we are instead interested in expansions of $(a-b)^n$, the triangle becomes:

$$
\begin{array}{c}
+1 \\
+1 \quad -1 \\
+1 \quad -2 \quad +1 \\
+1 \quad -3 \quad +3 \quad -1 \\
+1 \quad -4 \quad +6 \quad -4 \quad +1 \\
+1 \quad -5 \quad +10 \quad -10 \quad +5 \quad -1 \\
+1 \quad -6 \quad +15 \quad -20 \quad +15 \quad -6 \quad +1 \\
+1 \quad -7 \quad +21 \quad -35 \quad +35 \quad -21 \quad +7 \quad -1 \\
+1 \quad -8 \quad +28 \quad -56 \quad +70 \quad -56 \quad +28 \quad -8 \quad +1
\end{array}
$$

2.7 Problems

Problem 1

Write the prime factorization of:

$$32x^2y$$

Problem 2

Write the prime factorization of:

$$20y^3$$

Problem 3

Write the GCF and LCM of:

$$32x^2y \qquad 20y^3$$

Problem 4

Write the GCF and LCM of:

$$15x^2 \qquad 25x$$

Problem 5

Show that the GCF and LCM of

$$4a^2b \qquad 6ab \qquad 8ab^2$$

are:

$$GCF = 2 \times a \times b = 2ab$$
$$LCM = 2^3 \times 3 \times a^2 \times b^2 = 24a^2b^2$$

Problem 6

Let $x = 3$. Evaluate:

$$2 \times \left(1 + x^2\right)$$

Problem 7

Let $x = 2$ and $y = 3$. Evaluate:

$$-2^y \times ((5+5) - y^x) + 9$$

Problem 8

Use the FOIL method to evaluate:

$$(4+x)(4-5) \qquad (x+7)(2x-5) \qquad (-3x+4)(-2x-5)$$

Problem 9

Use the FOIL method to evaluate:

$$(x+y)^2 \qquad (x+y)(x-y)$$

Problem 10

Simplify:

$$\frac{12x}{3} \times \frac{3}{2} \div \frac{7}{4x} \qquad \frac{4x}{5} + \frac{7x}{9}$$

Problem 11

Simplify:

$$\frac{x}{4} \times \frac{2}{3x} \qquad \frac{4}{9x} + \frac{x}{3} - \frac{4}{18x}$$

Problem 12

Simplify:

$$\frac{4}{x-2} + \frac{6}{x-3} \qquad \frac{1}{x+1} + \frac{2}{x+6}$$

Problem 13

Simplify:

$$x^2 \cdot x^3 \cdot x^{-2} \qquad \sqrt{81} \times \sqrt{x^2}$$

Problem 14

Simplify:

$$\frac{x^3}{y^6} \cdot x^{-2} \cdot y^4 \qquad \left(x^2 \cdot y^2\right)^{1/3} \qquad \frac{1}{3}\sqrt{x^2} + \frac{2xy}{3\sqrt{y^2}}$$

Problem 15

Simplify:

$$\left(x^2 + 2y^2 + 3\right) + \left(-3x^2 - y^2 + 6\right)$$

Problem 16

Simplify:

$$2\left(-3z^2 + 8y^2\right) - 3\left(5z - 2y^2 - 3\right)$$

Problem 17

Simplify:

$$(y+4)(y-4) - \left(y^2 - 10\right)$$

Problem 18

Simplify:

$$(x-2)(3x+2y-2)$$

Problem 19

Simplify:

$$\left(2z - 3y^2 + 4\right)(z+3)$$

Problem 20

Simplify:

$$(2x+3y-4)(x-2y+1)$$

Problem 21

Simplify:

$$(a+b)(a-b)$$

Problem 22

Simplify:

$$(a + b)(c + d + e)$$

Problem 23

Simplify:

$$(a + b)^2 - (a - b)^2$$

3 Equations

Formally, an *equation* is made of two expressions joined by an equal sign. However 'ugly' an equation may appear, the left side and right side are equal. Examples of equations are:

$$
\begin{aligned}
5 + 5 &= 10 \\
3 - x &= 2 \\
x^2 &= 24 + 25
\end{aligned}
$$

3.1 Manipulating an Equation

Whatever manipulations are performed on an equation, the information it contains must not change: the left side must always equal the right side. This constrains us to not use equations to produce nonsense. The number of allowed manipulations boil down to just *two*: you may add zero to an equation, or multiply one into an equation, that's it. Below we explore a few special cases of either multiplying by one or adding zero.

Adding or Subtracting a Number

Adding or subtracting the same number on both sides of an equation is one allowed operation. For example, consider the equation

$$x - 5 = 3\ .$$

If we simply add 5 to each side, the equation becomes

$$x - 5 + 5 = 3 + 5\ ,$$

which simplifies very nicely:

$$x = 8$$

By isolating x, we have successfully 'solved for' its value, plugging $x - 8$ into the original equation, we find no mistakes:

$$8 - 5 = 3$$

Multiplying or Dividing a Number

Multiplying or dividing the same number on both sides of an equation is another allowed operation. For example, let us isolate x in the equation

$$2x = 24\ .$$

To do so, multiply both sides by a factor of 1/2 to get

$$\frac{1}{2} \times 2x = \frac{1}{2} \times 24\ ,$$

and the coefficients cancel out on the left:

$$x = 12$$

Raising to a Power

Raising each side to a power is yet another allowed operation on an equation. For example, consider the equation. For example, consider the equation:

$$\sqrt{x} = 3$$

If we raise each side to the power 2, the equation becomes

$$\left(\sqrt{x}\right)^2 = 3^2 \ ,$$

where the square root and the 2-power cancel, giving

$$\left(\sqrt{x}\right)^2 = \left(x^{1/2}\right)^2 = x^1 = x = 9 \ .$$

Taking a Root

Taking the square root (or cube root, etc.) of each side is one more allowed operation on an equation. For example, consider the equation:

$$x^2 = 49$$

Putting the square root symbol around the entire left side and the entire right side, the equation becomes

$$\sqrt{x^2} = \sqrt{49} \ ,$$

where the square root and the 2-power cancel, giving

$$\sqrt{x^2} = \left(x^2\right)^{1/2} = x^1 = x = 7 \ .$$

Before moving on however, note that $x = -7$ also satisfies the original equation $x^2 = 49$. This in fact true for all square roots: whenever we take the square root of a number, there are *two* results, one positive and one negative. In the most general case, we will always have

$$\sqrt{a^2} = \begin{cases} +a \\ -a \end{cases} \ ,$$

usually condensed using the 'plus-or-minus' symbol ($\pm$):

$$\sqrt{a^2} = \pm a$$

Strategy

When performing operations that change an equation (on both sides!), make sure to properly apply multiplied terms, divided terms, and exponents/roots across the whole expression. To illustrate, consider the equation

$$3x + 6 = 12y + 18 \ .$$

There are many ways to manipulate the equation above, but for no particular reason, let us try the square root operation. It would be WRONG to try:

$$\sqrt{3x} + \sqrt{6} = \sqrt{12y} + \sqrt{18}\ ,$$

as this approach forgets to treat the left side and right side equally. To proceed correctly, we must write

$$\sqrt{3x+6} = \sqrt{12y+18}\ .$$

If instead we wanted to square each side of the equation, it would be WRONG to try

$$3x^2 + 6^2 = 12y^2 + 18\ ,$$

which again forgets to treat each side equally. Instead, we must have

$$(3x+6)^2 = (12y+18)^2\ .$$

3.2 Solving for a Variable

When a variable is embedded in an equation, we often need to determine its exact value to solve a problem. This is done by manipulating the equation so as to 'get the variable by itself' on the left side or the right side, a process called *solving for a variable.*

Example 1

Solve for x:

$$\frac{4x}{5} + \frac{7x}{9} = 4$$

Step 1: Rewrite fractions with a common denominator:

$$\frac{36x}{45} + \frac{35x}{45} = 4$$

Step 2: Combine like terms:

$$\frac{71x}{45} = 4$$

Step 3: Multiply both sides by 45:

$$\frac{71x}{\cancel{45}} \times \cancel{45} = 4 \times 45$$

Step 4: Divide both sides by 71 and simplify:

$$x = \frac{4 \times 45}{71} = \frac{180}{71}$$

Example 2

Find two solutions for x:

$$x^2 + 3 = 12$$

Step 1: Subtract 3 from both sides:

$$x^2 = 9$$

Step 2: Take the square root of each side:

$$\sqrt{x^2} = \sqrt{9}$$

Step 3: Simplify to write the result:

$$x = \pm 3$$

Example 3

Solve for x:

$$6 = 2\sqrt{x+3}$$

Step 1: Divide by 2 on both sides:

$$3 = \sqrt{x+3}$$

Step 2: Raise each side to the power 2:

$$3^2 = \left(\sqrt{x+3}\right)^2$$

Step 3: Simplify:

$$9 = x + 3$$

Step 4: Subtract 3 from both sides:

$$6 = x$$

Example 4

If $y = 2$, determine all solutions for x:

$$x^2 - 10y - 5 = 0$$

Step 1: Add $10y + 5$ to both sides:

$$x^2 = 10y + 5$$

Step 2: Take the square root of both sides:

$$x = \pm\sqrt{10y + 5}$$

Step 3: Substitute $y = 2$ and simplify:

$$x = \pm\sqrt{25}$$

Step 4: Use the a table, calculator, or memory to write the final answer:

$$x = \pm 5$$

3.3 Systems of Equations

When handed multiple equations containing multiple variables, this is called a system of equations. The number of equations/variables is called the *order* of the system. For example, below we have an order-two system, having two equations and two variables:

$$4x + 2y = 20$$
$$x + 3y = 15$$

There is a reliable technique for solving a system of two equations and two variables. Using the above system as an example, the following procedure generally works:

- Solve either equation for x or y, whichever is easier.

$$y = 10 - 2x$$
$$x + 3y = 15$$

- Substitute the previous result into the unused equation.

$$x + 3\,(10 - 2x) = 15$$

- The resulting equation should contain only one variable:

$$x = 3$$

- Substitute the known variable value into either one of the equations.

$$3 + 3y = 15$$

- Solve for the last unknown:

$$y = 4$$

If the number of equations/variables does not match, the system is either *underdetermined* (not enough information), or *overdetermined* (conflicting information). In either case, there is no clear solution to the system.

3.4 Linear vs. Nonlinear Equations

A(n) (system of) equation(s) is considered *linear* if all variables x, y, z, etc. have an exponent of exactly *one*. A linear equation should never have terms such $\sqrt{x}$, or $x \cdot y$, or z^2, as these would be considered *nonlinear* terms. For example, the set of equations

$$4x + 2y = 20$$
$$x + 3y = 15$$

qualifies as an order-two linear system, and can always be solved by the procedure outlined above.

On the other hand the system

$$\begin{aligned} x^2 + y &= 21 \\ xy &= 20 \end{aligned}$$

contains two nonlinear equations due to the x^2- any xy-terms. Nonetheless, the solution can be attempted by reducing the system to having one equation and one variable. In this case, substitute $y = 20/x$ into the first equation and simplify to get

$$x^3 - 21x + 20 = 0\,,$$

which is now even *more* nonlinear, but *does* have a solution (it happens to be $x = 4$).

3.5 Transcendental Equations

An equation is *transcendental* if the variable(s) being solved for cannot be isolated by the usual tools of algebra. For instance, the equation

$$2^x = x^2$$

qualifies as transcendental, as no ordinary process can be used to isolate x. Many transcendental equations can be solved by analytical means, or when that fails, by computer.

3.6 Problems

Problem 1

Solve for x:

$$2x + 1 = -5$$

Problem 2

Solve for x:

$$-1.5x + 4.59 = 1.5x - 0.9$$

Problem 3

Solve for x:

$$\frac{x}{2} + 5 + \frac{x}{2} = \frac{-5x}{4} + 3\,(x - 1)$$

Problem 4

Solve for x:

$$\frac{2}{x} + \frac{5}{4} = \frac{3}{x}$$

Problem 5

Solve for x:

$$\frac{3}{x+1} + \frac{2}{x+1} = 1$$

Problem 6

Solve for x:

$$\sqrt{x} + 2 = 4$$

Problem 7

Solve for x:

$$10 = \sqrt{x} + 1$$

Problem 8

Solve for x:

$$28 = \sqrt{x-3} + 3$$

Problem 9

Solve for x if $y = 3$:

$$\sqrt{4x - y} = 5$$

Problem 10

Solve for x if $y = 2$:

$$1 = \sqrt{x+y} - 5$$

Problem 11

Solve for x:

$$\sqrt{5x+3} = \sqrt{4x+5}$$

Problem 12

Solve for x:

$$\sqrt[3]{x} + 4 = 6$$

Problem 13

Solve for x:

$$12 = 3 - \sqrt[3]{x+2}$$

Problem 14

The area of a circle is πR^2, where $\pi \approx 3.14$ and R is the radius (center-to-edge distance). If the area of a certain pizza is $100\,\text{in}^2$, calculate the length of a slice (same as R).

Problem 15

At temperature T in Celsius, the speed of sound v in air (in meters per second) is given by

$$v \approx 331.1 \times \sqrt{1 + \frac{T}{273.15}}\,.$$

If the speed of sound in a certain place is measure to be exactly $350\ m/s$, calculate the required temperature.

Problem 16

Solve the system for x and y:

$$x - 4y = -18 \qquad\qquad -x + 3y = 11$$

Problem 17

Solve the system for x and y:

$$3x + 8y = 15 \qquad\qquad 2x - 8y = 10$$

Problem 18

Solve the system for x and y:

$$-10x - 5y = 45 \qquad\qquad 10x - 3y = -5$$

Problem 19

Solve the system for x and y:

$$-4x - 3y = -11 \qquad\qquad -5x + 3y = -61$$

Problem 20

Solve the system for x and y:

$$-12x - 5y = 40 \qquad\qquad 12x - 11y = 88$$

Problem 21

Below is a long division with some digits hidden. Which of $(5, 6, 7, 8)$ could represent one of the empty boxes?

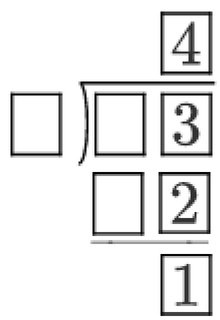


Figure 2: Long division with missing boxes.

Solution 21

Label the divisor as x, where y and z are integer digits. Observe $10x + 3 - (10z + 2) = 1$ to conclude $y = z$. Observe next that

$$\frac{10y + 3}{x} = 4 + \frac{1}{x} \; ,$$

which only delivers integer $y = 3$ if $x = 8$.

Problem 22

There is a triplet of positive numbers such that their sum and their product are equal, for example, $(1, 2, 3)$:

$$1 + 2 + 3 = 1 \times 2 \times 3$$

But, are there more triplets of positive numbers that have this property?

Solution 22

We want to check if any solutions to $x+y+z=xyz$ exist. Solve for z to get

$$z = \frac{x+y}{xy-1}\,,$$

corresponding to an infinite collection of points.

Problem 23

What three digits are represented by X, Y, and Z in the addition problem shown?

$$\begin{array}{r} \mathbf{XZY} \\ +\ \mathbf{XYZ} \\ \hline \mathbf{YZX} \end{array}$$

Figure 3: Incognito arithmetic problem.

Solution 23

The first column indicates $Y \neq 0$. The second column suggests $Z+Y \geq 10$, and furthermore $10 + 10Z + 10Y = 10Z + 100$, giving $Y = 9$. Thus, the first column is only satisfied by $X = 4$. Finally, the third column tells us $Y + Z = 10 + X$, or $Z = 5$.

Problem 24

Given $a = b$, spot the error in the following 'proof' that $2 = 1$:

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= b(a-b) \\ \cancel{(a-b)}(a+b) &= b\cancel{(a-b)} \\ a+b &= b \\ b+b &= b \\ 2b &= b \\ 2 &= 1 \end{aligned}$$

Solution 24

The cancellation of $(a-b)$ is equivalent to division by zero, which is illegal.

4 Rate, Interval, and Work

4.1 Rates

A *rate* is defined as some quantity, such as a cost, weight, or area, measured against another quantity, such as time or distance. Rates are *always* ratios, containing two pieces of information, hence the similarity in name. Common examples of rates may occur as:

$$14\,\frac{dollar}{hour} \qquad\qquad 15\,\frac{meter}{second} \qquad\qquad 55\,\frac{pound}{inch^2}$$

In general, we shall denote a rate with the variable letter R. For instance, for a car traveling at 40 miles per hour (abbreviated mi/h), we may write

$$R = 40\,\frac{mi}{h}\ .$$

4.2 Units and Dimension

Quantities such as dollars, seconds, or pounds are called *units*, also called the *dimension* of a number. Unit quantities can be treated identically as variables, and are subject to the same rules. For instance, we may add numbers carrying units by combining like terms as in

$$2\,meter + 3\,meter = 5\,meter\ ,$$

or multiply dimensional numbers such as

$$2\,second \times 3\,second = 6\,second^2\ .$$

As long as we obey the rules for combining fractions, rates can also be combined, as in

$$14\,\frac{dollar}{hour} - 3\,\frac{dollar}{hour} = 11\,\frac{dollar}{hour}\ .$$

To remove any ambiguity, the above calculations are identical to

$$\begin{aligned} 2x + 3x &= 5x \\ 2y \times 3y &= 6y^2 \\ 14z - 3z &= 11z\ , \end{aligned}$$

where x replaces $meter$, y replaces $second$, and z replaces $dollar/hour$. Any quantities that have no units, i.e. plain real numbers, are called *dimensionless*.

4.3 Interval

Looking again at the definition of a rate, each has a numerator of a certain dimension (dollars, meters, etc.), along with a denominator of a different dimension (usually *time*, but not always.)

For any given rate R, we can create a new number called an *interval*, denoted I, having units matching those in the denominator of R. For instance, consider an example rate

$$R = 11\,\frac{dollar}{hour}\ .$$

To construct an interval, choose *some* quantity I measured specifically in hours, perhaps

$$I = 3\,hour\ .$$

By computing the product $R \times I$, we find

$$R \times I = 11\,\frac{dollar}{\cancel{hour}} \times 3\,\cancel{hour} = 33\,dollar\ ,$$

which cancels the 'hours' unit altogether, leaving the result in only dollars. That is, the product $R \times I$ can have any numerical value, but must have units matching those in the numerator of R.

4.4 Work

The product of a rate and an interval (of proper dimension) shall be called *work*, denoted W:

$$W = R \times I\ .$$

Reiterating the previous example, we say that a rate of $R = 11\,dollar/hour$ multiplied by an interval of $I = 3\,hour$ amounts to the work value of $W = 33\,dollar$.

The so-called 'work equation' can be solved for R, namely

$$R = \frac{W}{I}\ ,$$

telling us that a rate can be interpreted as the work value divided by the interval. Similarly, the interval

$$I = \frac{W}{R}$$

is the ratio of the work value to the rate.

Example 1

Joseph works as a journeyman electrician for a pay rate of 30 dollars per hour. After a 9-hour workday, what are his daily earnings?

Step 1: Identify the rate, the interval, and the work value:

$$R = 30\,\frac{dollar}{hour} \qquad I = 9\,hour \qquad W = ?$$

Step 2: Apply the version of the work equation with W as the unknown:

$$W = R \times I = 30\,\frac{dollar}{hour} \times 9\,hour$$

Step 3: Cancel the *hour* units, and simplify the fraction:

$$W = R \times I = 30\,\frac{dollar}{\cancel{hour}} \times 9\,\cancel{hour} = 30 \times 9\,dollar = 270\,dollar$$

Example 2

Joseph keeps working as a journeyman electrician for a pay rate of 30 dollars per hour. After a long workday, his earnings were 330 dollars. How many hours did he work?

Step 1: Identify the rate, the interval, and the work value:

$$R = 30\,\frac{dollar}{hour} \qquad I = ? \qquad W = 330\,dollar$$

Step 2: Apply the version of the work equation with I as the unknown:

$$I = \frac{W}{R} = \frac{330\,dollar}{30\,dollar/hour}$$

Step 3: Cancel the *dollar* units, and simplify the fraction:

$$I = \frac{W}{R} = \frac{330\,\cancel{dollar}}{30\,\cancel{dollar}/hour} = \frac{330}{30}\,hour = 11\,hour$$

Example 3

Joseph agrees to work on a special project for 10 hours. His earnings for the project were 350 dollars. Calculate the pay rate for the special project.

Step 1: Identify the rate, the interval, and the work value:

$$R = ? \qquad I = 10\,hour \qquad W = 350\,dollar$$

Step 2: Apply the version of the work equation with R as the unknown:

$$R = \frac{W}{I} = \frac{350\,dollar}{10\,hour}$$

Step 3: Simplify to write the answer:

$$R = 35\,\frac{dollar}{hour}$$

Example 4

A bee can visit 644 flowers in 7 hours. How many flowers can the bee visit in 9 hours?

Step 1: Set up a work equation containing known information:

$$W = R \times I$$
$$644\,flower = R \times 7\,hour$$

Step 2: Solve for the rate R:

$$R = \frac{644}{7} \frac{flower}{hour} = 92 \frac{flower}{hour}$$

Step 3: Step up a work equation with R known and $I = 9\,hour$ to get the result:

$$\begin{aligned} W &= R \times I \\ W &= 92 \frac{flower}{\cancel{hour}} \times 9\,\cancel{hour} \\ W &= 92 \times 9\,flower = 828\,flower \end{aligned}$$

<u>Example 5</u>

A certain brand of concentrated iced tea advises that 2 *oz* of product mixed with 12 *oz* of water makes the perfect serving. How much of each ingredient is needed to fill a 64 *oz* container at the proper ratio?

Step 1: Write an equation containing known information:

$$14\,oz\ tea = 2\,oz\ product + 12\,oz\ water$$

Step 2: Identify the following rate:

$$R = \frac{14\,oz\ tea}{2\,oz\ product + 12\,oz\ water}$$

Step 3: Set up a work equation with the interval I as the unknown:

$$\begin{aligned} I &= \frac{W}{R} = W \times \frac{1}{R} \\ I &= 64\,oz\ tea \times \left(\frac{2\,oz\ product + 12\,oz\ water}{14\,oz\ tea} \right) \end{aligned}$$

Step 4: Simplify:

$$\begin{aligned} I &= 64\,\cancel{oz\ tea} \times \left(\frac{2\,oz\ product + 12\,oz\ water}{14\,\cancel{oz\ tea}} \right) \\ I &= \frac{128}{14}\,oz\ product + \frac{768}{14}\,oz\ water \\ I &= 9.14\,oz\ product + 54.86\,oz\ water \end{aligned}$$

4.5 Effective Rate

Certain situations call for individual rates to be combined into an *effective rate*. That is, if we have a handful of compatible rates R_1, R_2, etc., the effective rate is

$$R_{eff} = R_1 + R_2 + \cdots\ ,$$

obeying the same work equation

$$W = R_{eff} \times I \; .$$

Example 6

An inlet pipe can fill a swimming pool in 5 hours, while an outlet pipe can drain the pool in 8 hours. By mistake, a maintenance worker left both pipes open. Will the pool overflow?

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned} R &= R_{eff} = R_{fill} + R_{drain} \\ I &= ? \\ W &= 1\,pool \end{aligned}$$

Step 2: Determine the effective rate: (Hint: Find the LCM of 5, and 8.)

$$\begin{aligned} R_{eff} &= \frac{1\,pool}{5\,hour} - \frac{1\,pool}{8\,hour} \\ R_{eff} &= \frac{8\,pool}{40\,hour} - \frac{5\,pool}{40\,hour} \\ R_{eff} &= \frac{3\,pool}{40\,hour} = 0.075\,\frac{pool}{hour} \end{aligned}$$

Step 3: Observe that R_{eff} came out positive, meaning the pool will overflow.

Example 7

A cold water faucet can fill a bathtub in 12 minutes, and a hot water faucet can fill the bathtub in 18 minutes. The drain can empty the bathtub in 24 minutes. If both faucets are on and the drain is open, how long would it take to fill the bathtub?

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned} R &= R_{eff} = R_{hot} + R_{cold} + R_{drain} \\ I &= ? \\ W &= 1\,tub \end{aligned}$$

Step 2: Determine the effective rate: (Hint: Find the LCM of 12, 18, and 24.)

$$\begin{aligned} R_{eff} &= \frac{1\,tub}{12\,minute} + \frac{1\,tub}{18\,minute} - \frac{1\,tub}{24\,minute} \\ R_{eff} &= \frac{6\,tub}{72\,minute} + \frac{4\,tub}{72\,minute} - \frac{3\,tub}{24\,minute} \\ R_{eff} &= \frac{7\,tub}{72\,minute} \end{aligned}$$

Step 3: Apply the version of the work equation with I as the unknown:

$$I = \frac{W}{R_{eff}} = W \times \frac{1}{R_{eff}} = 1\,tub \times \frac{72\,minute}{7\,tub}$$

Step 4: Cancel the *tub* units, and simplify the fraction:

$$I = 1\,\cancel{tub} \times \frac{72\,minute}{7\,\cancel{tub}} = \frac{72}{7}\,minute = 10.29\,minute$$

Example 8

It takes Tom 4 hours to build a fence. If he hires Jack to help him, together they can do the job in just 3 hours. If Jack built the same fence alone, how long would it take him?

Step 1: Discern the following equations from the problem statement:

$$\begin{aligned} 1\,fence &= R_{Tom} \times 4\,hour \\ 1\,fence &= (R_{Tom} + R_{Jack}) \times 3\,hour \\ 1\,fence &= R_{Jack} \times I_{Jack} \end{aligned}$$

Step 2: Solve the middle equation for R_{Jack}:

$$R_{Jack} = \frac{1\,fence}{3\,hour} - R_{Tom} = \frac{1\,fence}{3\,hour} - \frac{1\,fence}{4\,hour} = \frac{1\,fence}{12\,hour}$$

Step 3: Solve the third equation for I_{Jack}:

$$I_{Jack} = \frac{1\,fence}{R_{Jack}} = 1\,\cancel{fence} \times \frac{12\,hour}{1\,\cancel{fence}} = 12\,hour$$

Example 9

A woodworking shop is open eight hours per day. Tony can build three birdhouses in the eight-hour workday. Joe is new, and accidentally destroys one birdhouse every four hours. If the shop hires a third builder for six hours per day, how many birdhouses must he build during his shift in order for the shop to produce five birdhouses in an eight-hour day? Hint: Let x be the number of birdhouses the new worker builds per six hours.

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned} R &= R_{eff} = R_{Tony} + R_{Joe} + \frac{x\,house}{6\,hour} \\ I &= 8\,hour \\ W &= 5\,house \end{aligned}$$

Step 2: Write the work equation using the information provided and simplify:

$$\begin{aligned} W &= R_{eff} \times I \\ 5\,house &= \left(\frac{3\,house}{8\,hour} - \frac{1\,house}{4\,hour} + \frac{x\,house}{6\,hour}\right) \times 8\,hour \\ 5\,\cancel{house} &= \left(\frac{3\,\cancel{house}}{8\,\cancel{hour}} - \frac{1\,\cancel{house}}{4\,\cancel{hour}} + \frac{x\,\cancel{house}}{6\,\cancel{hour}}\right) \times 8\,\cancel{hour} \\ 5 &= \left(\frac{3}{8} - \frac{1}{4} + \frac{x}{6}\right) \times 8 \end{aligned}$$

Step 3: Solve the above for x:

$$x = 6 \cdot \left(\frac{5}{8} - \frac{3}{8} + \frac{1}{4}\right) = 3$$

5 Graphing

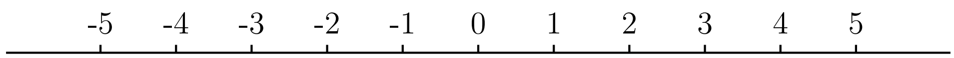
Graphing is the 'art' of projecting mathematical information onto a visual or other sensory medium.

5.1 One-Dimensional Graphs

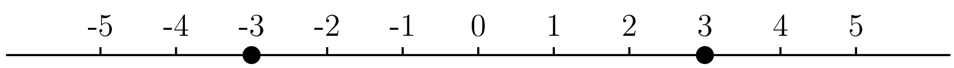
By now, we are familiar with solving a single equation containing a single variable. Taking the example equation

$$x^2 = 9 ,$$

we may easily verify two solutions $x = 3$ and $x = -3$. In order to *visualize* these solutions, we prepare a blank number line as follows:



On the number line, the solutions $x = \pm 3$ are easily marked, or *plotted* at their respective locations:



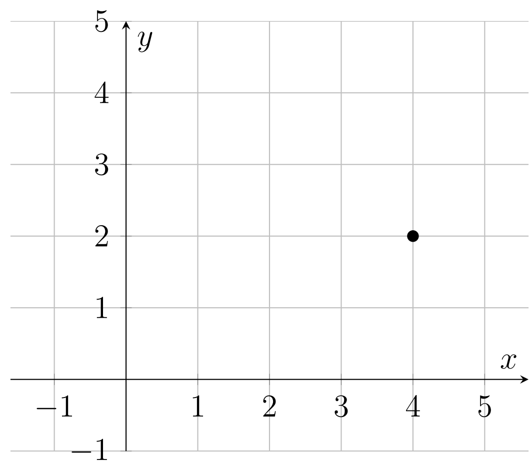
Used for this purpose, the real number line is a *one-dimensional* graph. Any particular location on the graph is called a *point*.

5.2 Two-Dimensional Graphs

We can also solve multiple equations containing multiple variables. For instance, the order-two system

$$\begin{aligned} 2x + 3y &= 14 \\ x - y &= 2 \end{aligned}$$

is only satisfied by $x = 4$, $y = 2$. To visualize these two solutions though, it only makes sense to have a separate graph for the x-solution, and another graph for the y-solution. To do so, we draw a second number line perpendicular to the familiar one-dimensional number line. Labeling each axis as x and y respectively, the solution to the above system appears as a single point in the following graph:



In the above, the marked location $x = 2$, $y = 4$ denotes the solution to the solution to the system of equations.

5.3 Ordered-Pair Solutions

Suppose you are given *one* equation containing *two* variables, such as

$$4y - 3x = -2\ .$$

To 'solve' the (underdetermined) equation, surely we can solve for x, but that still leaves y unknown (and vice-versa). Doing so *anyway*, we have

$$x = \frac{4y + 2}{3}\ .$$

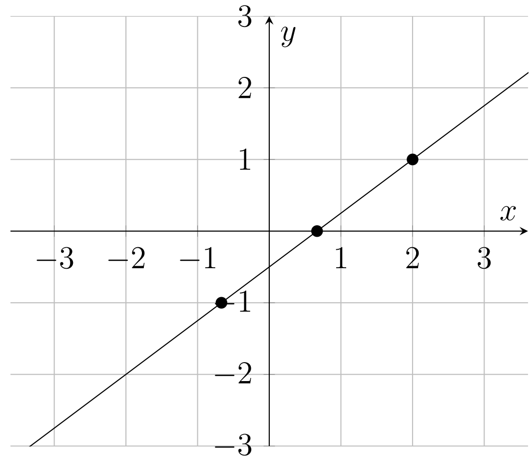
Since x and y are both variable, we have freedom to play with *one* of their values, which will strictly determine the other. For instance, setting $y = 0$, we quickly find that $x = 2/3$. Choosing instead $y = -1$, the corresponding x-value must be $x = -2/3$. Similarly, $y = 1$ gives $x = 2$, and so on.

The groups of numbers $(x = 2/3, y = 0)$, $(x = -2/3, y = -1)$, $(x = 2, y = 1)$, etc. are called *ordered pair solutions* to the equation. Often, the $x =$, $y =$ nomenclature is omitted by convention unless explicitly needed. Apart from

$$(2/3, 0) \qquad\qquad (-2/3, -1) \qquad\qquad (2, 1)\ ,$$

it turns out there are an infinite number of ordered-pair solutions to the equation $4y - 3x = -2$. For any x we could possibly choose, there is some corresponding y that satisfies the equation. This is not to say that *any* pair of numbers is a solution to the equation. For instance, the choice $x = 0$, $y = 0$ leads to the untrue statement $0 = 2/3$.

Now we shall plot the known ordered-pair solutions to $4y - 3x = -2$. Plotted on a two-dimensional graph, the three points seem to adhere to a straight line. Extending that line through the points and beyond, the result comes out as follows:



The straight line connecting the known points in fact runs through *all* solutions to the equation $4y - 3x = -2$. This is precisely why such an equation is classified as 'linear', as its solution 'looks like' a straight line on a two-dimensional graph.

For another example, consider the equation

$$x^2 + y = 3\ .$$

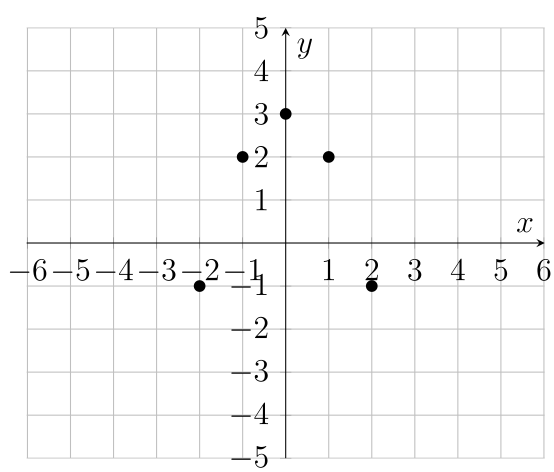
Solutions are most easily found by solving for y:

$$y = 3 - x^2$$

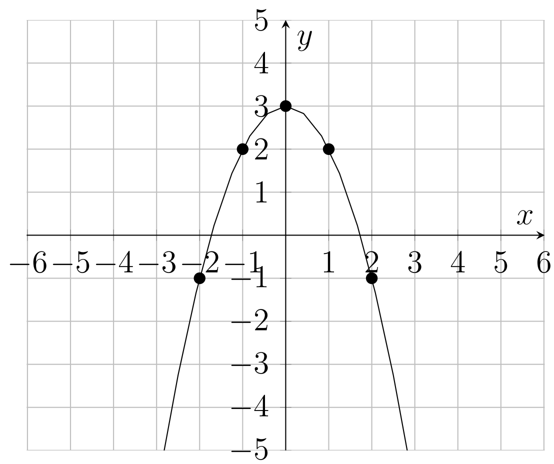
Choosing a few x-values, we easily generate the following ordered-pairs:

$(-2, -1)$ $\qquad$ $(-1, 2)$ $\qquad$ $(0, 3)$ $\qquad$ $(1, 2)$ $\qquad$ $(2, -1)$

Plotting these on a two-dimensional graph, we find:



Clearly, a straight line cannot be used to connect the points as drawn, thus $x^2 + y = 3$ is a nonlinear equation. The shape that *does* connect the points is called a *parabola*. Indeed, solutions to the equation on hand are 'parabolic' in form as shown:

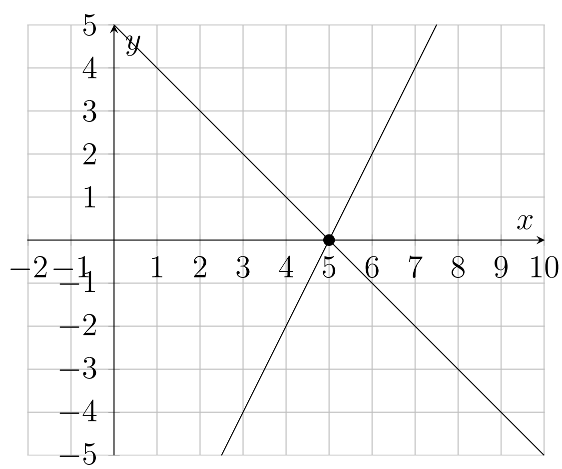


5.4 Graphical Solution Methods

Consider the system of linear equations of two equations and two unknowns:

$$x + y - 5 = 0 \qquad\qquad y - 2x + 10 = 0$$

While we may proceed analytically to solve the system, here we instead develop a 'shortcut' that involves plotting solutions to each equation on the same graph as follows:

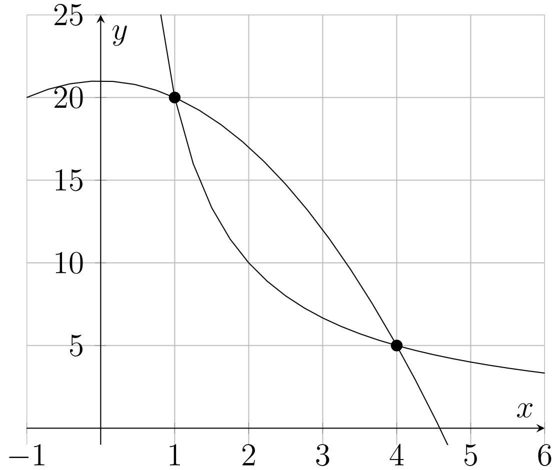


The two lines intersect at the point $(5, 0)$, which is precisely the same result one would attain analytically. Indeed, the original set of equations is satisfied only by $x = 5$, $y = 0$.

The graphical method also helps to solve nonlinear equations. Taking the system

$$x^2 + y = 21$$
$$xy = 20 \; ,$$

one quickly notices that attaining an analytic solution becomes a lengthy endeavor. Supposing we generate ordered-pair solutions to each equation, their combined plot appears as:



In the above, we observe the the intersection of a parabola, along with a special case of the general shape $xy =$ constant, called a *hyperbola*. Evidently, the system of equations is solved by two ordered pairs

$$(1, 20) \qquad\qquad (4, 5) \; .$$

In general, the number of solutions to a nonlinear system is not obvious. It's possible for one, two, zero, or even infinite solutions to exist.

Finally, we can apply the graphical method to a *single* equation, or even a transcendental equation. To illustrate, consider the nonlinear equation

$$x^2 - x - 6 = 0 \; ,$$

which in general may or may not have valid solutions. To proceed, move the x^2-term to the other side of the equation, giving

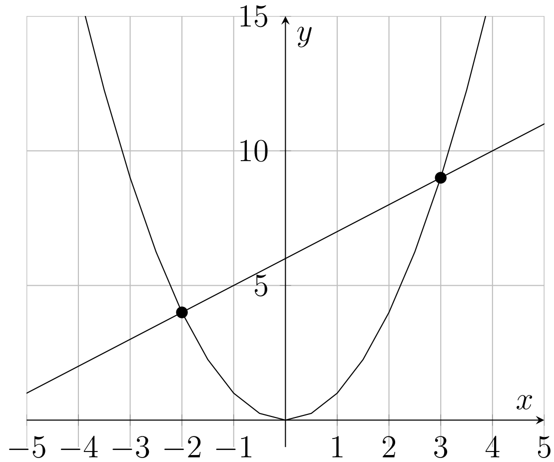
$$x^2 = x + 6 \; .$$

Since the left side and the right side are equal, we are allowed to break the equation into two equal parts:

$$y = x^2 \qquad\qquad y = x + 6$$

If in doubt, simply eliminate y to restore the original equation.

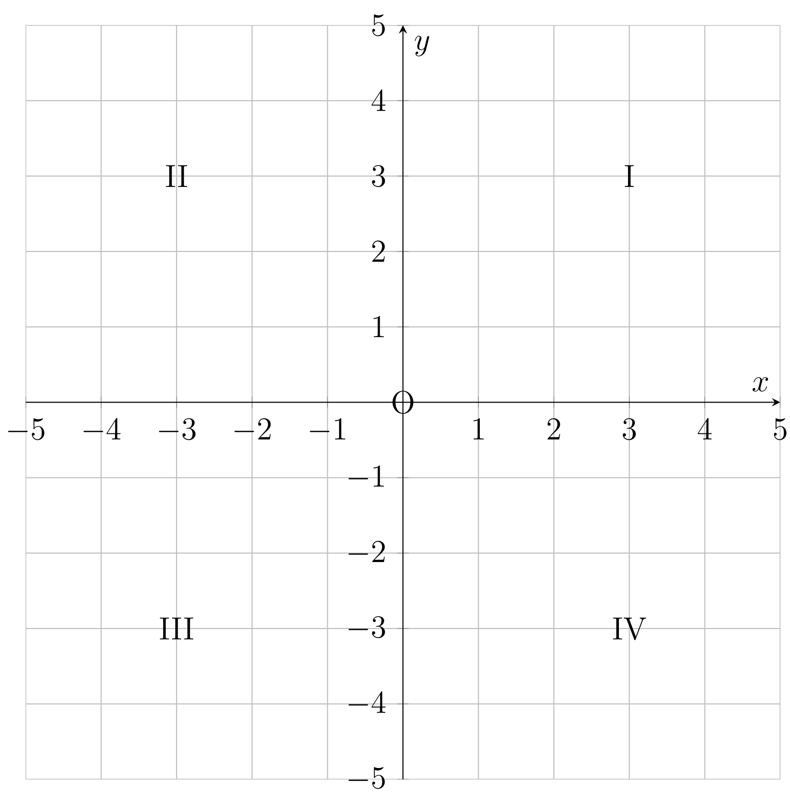
Next, plot the ordered-pair solutions to each (individual) equation, generating the following graph:



The two plots intersect at $x = -2$, and also at $x = 3$. Evidently, the equation $x^2 - x - 6 = 0$ has two solutions at these points.

5.5 Cartesian Plane

The two-dimensional grid formed by perpendicular number lines is called the *Cartesian plane*. By convention, the number lines intersect at position $x = 0, y = 0$ as shown:



Quadrants

- The quarter-plane with $x > 0$, $y > 0$ is the *first quadrant* (I).

- The quarter-plane with $x < 0$, $y > 0$ is the *second quadrant* (II).
- The quarter-plane with $x < 0$, $y < 0$ is the *third quadrant* (III).
- The quarter-plane with $x > 0$, $y < 0$ is the *fourth quadrant* (IV).

Terminology

- The horizontal number line is called the *x-axis*.
- The vertical number line is called the *y-axis*.
- The point $x = 0$, $y = 0$ is called the *origin*.
- The Cartesian plane extends to $x = \pm\infty$ and $y = \pm\infty$.

6 Straight Line Analysis

6.1 General Equation

Equations containing only linear terms, i.e. nothing like x^2, $\sqrt{y}$, etc., follow a general form

$$A + Bx + Cy + Dz + \cdots = 0\,,$$

where A, B, C, etc., are numerical coefficients. Taking the special case of two dimensions, the above reduces to

$$A + Bx + Cy = 0\,,$$

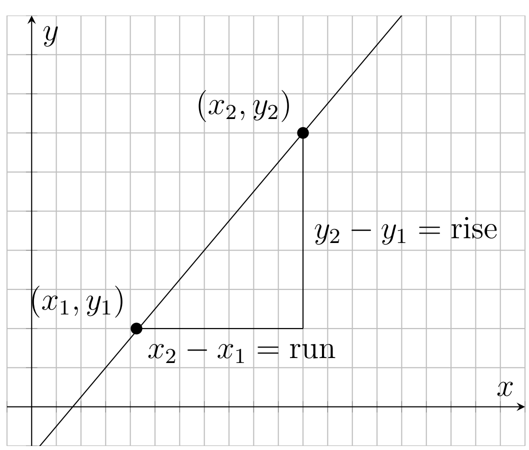
which we call the 'equation of a line' in the Cartesian plane. The position and orientation of the line is contained in the coefficients A, B, and C. All ordered pairs (x, y) that solve the above equation are points on the line.

6.2 Slope

The formal name for the 'orientation' of a line is called the *slope*. Taking any two ordered-pair solutions on the line, call them (x_1, y_1), (x_2, y_2), the slope of the line is defined as

$$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}\,.$$

That is, the slope of a straight line is the change in y divided by the change in x between any two points as shown:



A line with zero slope is strictly horizontal, as there is no change in y for any given x. When the slope is positive, the height y of the line is increasing for increasing x. By contrast, a negative slope means the height y is ramping downward for increasing x. Any vertical line has *infinite* slope, as all possible y-values correspond to the same x, and the slope equation gives division by zero.

Slope Analysis

Suppose we are given two points (x_1, y_1), (x_2, y_2) in the Cartesian plane that are known to fall onto a straight line. Let us take the general equation of a line, and apply it each point:

$$A + Bx_1 + Cy_1 = 0 \qquad\qquad A + Bx_2 + Cy_2 = 0$$

What we have is two equations and three unknowns, which is an overdetermined system - there is one variable too many. To proceed, divide each equation by C to get

$$\frac{A}{C} + \frac{B}{C}\,x_1 + y_1 = 0 \qquad\qquad \frac{A}{C} + \frac{B}{C}\,x_2 + y_2 = 0\,.$$

Since A, B and C are arbitrarily-valued, it does no hard to define

$$\frac{A}{C} = \tilde{A} \qquad\qquad \frac{B}{C} = \tilde{B}\,,$$

which effectively reduces the number of unknowns to two. (Don't forget that each x and y is known in this case.)

$$\tilde{A} + \tilde{B}\,x_1 + y_1 = 0 \qquad\qquad \tilde{A} + \tilde{B}\,x_2 + y_2 = 0\,.$$

Next, solve each equation for $\tilde{A}$ and set them equal:

$$\tilde{B}\,x_1 + y_1 = \tilde{B}\,x_2 + y_2$$

Condense all x-terms on one side, and all y-terms on the other:

$$\tilde{B}\,(x_1 - x_2) = y_2 - y_1$$

Finally, divide each side by $x_2 - x_1$ to land at

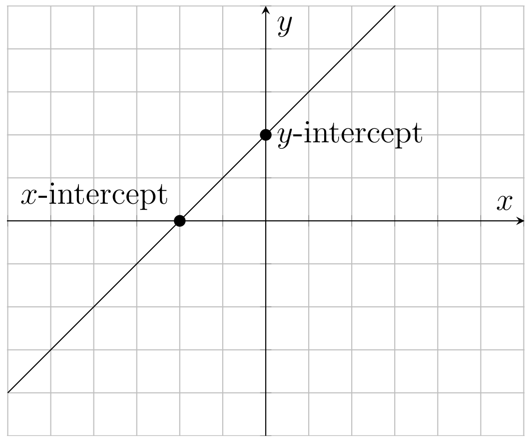
$$-\tilde{B} = \frac{y_2 - y_1}{x_2 - x_1}\,.$$

Comparing this result to the definition of the slope of a line, we find that $-\tilde{B}$ is precisely equal to the slope of the line. Restoring the original variables, we find

$$m = -\tilde{B} = -\frac{B}{C}\,.$$

6.3 Intercepts

Any straight line that is not purely vertical or not purely horizontal eventually hits both the x-axis and the y-axis. The location $(x_{int}, 0)$ where the line hits the x-axis is called the *x-intercept*. Similarly, the location $(0, y_{int})$ where the line hits the y-axis is called the *y-intercept* as shown:



Starting with the general equation $A+Bx+Cy=0$, the x- and y-intercepts, respectively, are determined by setting $y=0$ and $x=0$, in that order, giving

$$A + B\cdot 0 + C\cdot y_{int} = 0 \qquad\qquad A + B\cdot x_{int} + C\cdot 0 = 0\ ,$$

telling us

$$y_{int} = -\frac{A}{C} \qquad\qquad x_{int} = -\frac{A}{B}\ .$$

The values y_{int}, x_{int} combine to tell us the slope of the line by eliminating $-A$ between both equations:

$$C\cdot y_{int} = B\cdot x_{int} \qquad \to \qquad \frac{y_{int}}{x_{int}} = \frac{B}{C} = -m$$

6.4 Slope-Intercept Equation

Let us return to the general equation of a line in the Cartesian plane,

$$A + Bx + Cy = 0\ ,$$

and solve the equation for y:

$$y = \frac{-A - Bx}{C} = -\frac{A}{C} - \frac{B}{C}\,x\ .$$

Next, replace $-A/C$ and $-B/C$ with the equivalent terms y_{int}, m, respectively:

$$y = y_{int} + mx$$

By convention, the term y_{int} is almost always denoted as b, and written after the mx term. Finally, we have the *slope-intercept* form of the equation of a line:

$$y = mx + b$$

The variables m (slope) and b (y-intercept) contain *all* of the information needed to plot the entire line (that is, to find all ordered-pair solutions to $y = mx + b$).

6.5 Point-Slope Equation

Let us return to the equation that defines the slope of a line, namely

$$m = \frac{y_2 - y_1}{x_2 - x_1} \,.$$

Next, suppose that m has a known value, and that some point (x_1, y_1) on the line is also known. Letting (x, y) denote *any* other point on the line, we arrive at the *point-slope* equation of a line:

$$m\,(x - x_1) = y - y_1$$

As a sanity check, we can make sure that the point-slope form and the slope-intercept forms are equivalent. Solving the above for y, we have

$$y = mx - mx_1 + y_1 \,,$$

which is only true if $-mx_1 + y_1$ equals the y-intercept, or

$$y_1 = mx_1 + b \,,$$

which is precisely the slope-intercept equation as applied to (x_1, y_1).

6.6 Summary

In summary, we have developed three equations of a straight line:

- General form:
$$A + Bx + Cy = 0$$
- Slope-Intercept form:
$$y = mx + b$$
- Point-Slope form:
$$m\,(x - x_1) = y - y_1$$

The coefficients A, B, C contain information on the slope, the x-intercept, and the y-intercept. Explicitly, we found:

$$m = -\frac{B}{C} = \frac{y_2 - y_1}{x_2 - x_1} \qquad\qquad x_{int} = -\frac{A}{B} \qquad\qquad b = y_{int} = -\frac{A}{C}$$

6.7 Problems

Problem 1

Determine the slope of a line connecting the points $(2, 3)$ and $(7, 4)$.

Problem 2

Write the point-slope equation of a line connecting the points $(-1, 10)$ and $(5, 2)$.

Problem 3

Write the slope-intercept equation of a line connecting the points $(-8, -8)$ and $(-7, 9)$.

Problem 4

Find the y-intercept of a line having $m = 2$ that passes through $(2.5, 0)$.

Problem 5

Write the equation of a line with an x-intercept of 3, and a y-intercept of 6.

7 Quadratic Equations

7.1 Building the Form

Recall that the equation of a straight line in the Cartesian plane is given by

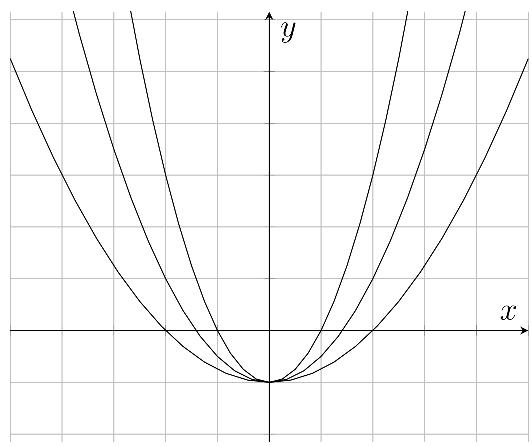
$$y = mx + b\,.$$

Any point (x, y) on the line is an ordered-pair solution to the equation $y = mx + b$, and the character of the line is given by its slope m and y-intercept b.

Let us now modify the straight line equation to replace x with a *nonlinear* term, particularly x^2. The y-intercept plays the same role, but by convention, we relabel $b \to k$. The notion of 'slope' becomes ambiguous, so let m be replaced by a general 'scaling factor' called a. To visualize this, we plot ordered-pair solutions to the equation

$$y = ax^2 + k$$

and momentarily fix k while allowing a to vary, generating the following graph:



In the above, observe that larger a-values lead to a 'skinny' parabolic curve, whereas smaller a-values widen the plot into a 'bowl' shape. Negative values of a would flip the plots upside-down.

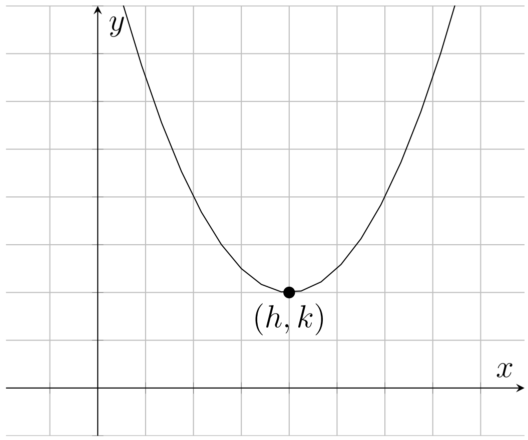
We can also translate the entire plot in the right/left direction by shifting the x-variable via

$$x \to x - h\,,$$

leading to the general equation of a shifted parabola

$$y = a\,(x - h)^2 + k\,,$$

sketched below:



In the above, observe that positive values of k shift the plot to the right of the y-axis. The point (h, k) is called the *vertex* of the parabola. The vertical line passing through the vertex is called the *axis*.

7.2 General Form

Another name for a scaled-and-shifted parabola is a *quadratic* equation. Proceeding by analogy to straight line analysis, observe that our quadratic equation

$$y = a\,(x-h)^2 + k$$

can also be written in general form, namely

$$y = Ax^2 + Bx + C\ ,$$

where the task on hand is to relate the new A, B, C to the scaling and shifting variables a, h, k. This is started by multiplying out the $(x-h)^2$-term in the top equation, and then combining like terms:

$$y = (a)\,x^2 + (-2ah)\,x + \left(ah^2 + k\right)$$

By comparing the coefficients on the x^2, x, and dimensionless terms, we find:

$$\begin{aligned} A &= a \\ B &= -2ah \\ C &= ah^2 + k \end{aligned}$$

Solving the system for a, h, and k, we have:

$$\begin{aligned} a &= A \\ h &= -\frac{B}{2A} \\ k &= C - \frac{B^2}{4A} \end{aligned}$$

7.3 Solving Quadratic Equations

The quadratic equation

$$y = a\,(x-h)^2 + k$$

can be solved for x with relative ease, remembering that square roots have both positive and negative solutions:

$$x = h \pm \sqrt{\frac{y-k}{a}}$$

It's important to notice that any y-value less than k cannot possibly be a valid solution, as no part of the parabola exists for $y < k$ (for positive a). If we try inserting $y < k$ anyway, the square root term contains a negative number, which is in fact *imaginary* (technical term). At the limit case $y = k$, the only solution corresponds to $x = h$.

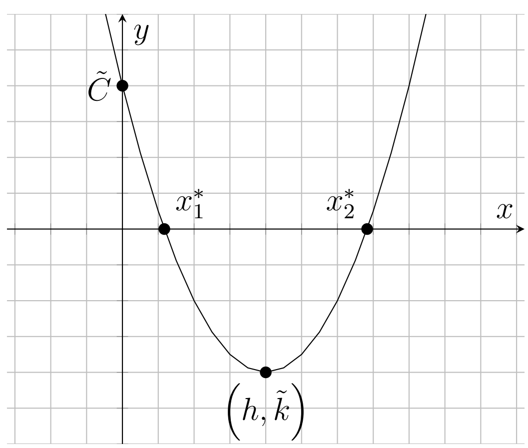
For a fixed 'height' y^* on the parabola, having solution(s) x^*, the governing equation is

$$y^* = a\,(x^*-h)^2 + k\;.$$

Since y^* and k are like terms (not multiplied by x or x^2), it's harmless to combine them into a new dimensionless constant $\tilde{k} = k - y^*$ such that

$$0 = a\,(x^*-h)^2 + \tilde{k}\;,$$

which transforms the problem into finding the x-intercept(s) of a shifted parabola with a vertex at $x = h$, $y = \tilde{k}$:



The substitution $\tilde{k} = k - y^*$ applies similarly in the general equation

$$y^* = A\,(x^*)^2 + Bx^* + C\;,$$

where by letting

$$\tilde{C} = C - y^* = ah^2 + k - y^* = ah^2 + \tilde{k}\;,$$

the problem similarly reduces to finding the x-intercept(s) of

$$0 = A\,(x^*)^2 + Bx^* + \tilde{C}\;.$$

We've shown that finding solutions to the quadratic equation can be reduced to the task of finding the x-intercept(s) of a shifted version of the equation. Since it would be burdensome to constantly maintain the superscripts in x^*, y^*, along with the tilde marks on $\tilde{k}$, $\tilde{C}$, let us for now on ignore these special marks, knowing they can be restored if needed. Thus, we can move forward by seeking solutions to:

$$0 = a\,(x-h)^2 + k \qquad \leftrightarrow \qquad 0 = Ax^2 + Bx + C$$

7.4 Quadratic Formula

Solving for x in the equation $0 = a\,(x-h)^2 + k$, we quickly find

$$x = h \pm \sqrt{-\frac{k}{a}}\ .$$

Replacing a, h, and k with their equivalent representations in terms of A, B and C, we have

$$x = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}\ ,$$

known as the (infamous) *quadratic formula.*

Explicitly, the quadratic formula gives solutions to

$$0 = Ax^2 + Bx + C\ .$$

In order for solutions to make sense, the quantity $B^2 - 4AC$ must evaluate to a positive number, otherwise the square root receives a negative input, which mean there are no real solutions to the equation. In the special case $B^2 = 4AC$, the only surviving solution is $x = -B/2A$.

7.5 Features of Quadratic Equations

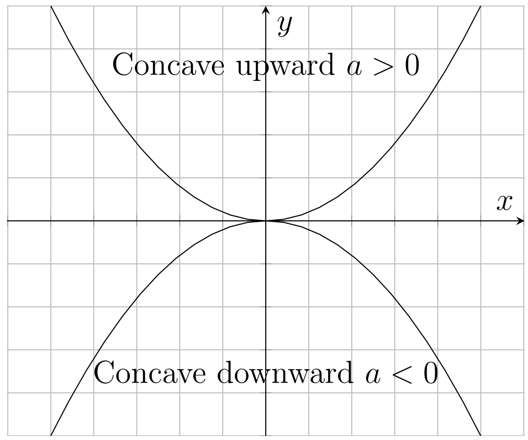
Supposing you are handed either of

$$y = a\,(x-h)^2 + k \qquad\qquad y = Ax^2 + Bx + C\ ,$$

a plot of ordered-pair solutions will appear as a parabola centered *somewhere* in the Cartesian plane, and will be *concave upward* (a bowl), or *concave downward* (a mountain).

Concavity

The concavity of a parabola is given directly by the A-term, or equivalently, the a-term. Upward concavity corresponds to positive values, where downward concavity corresponds to negative vales as shown:



Vertex

Regardless of its concavity, the vertex of the parabola is the location (h, k). The particular values h, k are especially obvious in $y = a\,(x-h)^2 + k$, but we dealing with the general form $y = Ax^2 + Bx + C$, we have to remember

$$h = -\frac{B}{2A} \qquad\qquad k = C - \frac{B^2}{4A} \, .$$

Axis

The axis of a parabola is the vertical line that slices down the middle, positioned at

$$x_h = h = -\frac{B}{2A} \, .$$

y-Intercept

A concave upward or concave downward parabola will *always* have a y-intercept, which is given by $x = 0$ in either equation

$$y_{int} = a\,(0-h)^2 + k = ah^2 + k \qquad\qquad y_{int} = A \cdot 0^2 + B \cdot 0 + C = C \, ,$$

telling us

$$y_{int} = C = ah^2 + k \, .$$

x-Intercept(s)

In general, a quadratic equation may have up to two x-intercepts, but as few as zero. Setting $y = 0$ in either version of the quadratic equation, we have

$$0 = a\,(x-h)^2 + k \qquad\qquad 0 = Ax^2 + Bx + C \, ,$$

with each being solved by the quadratic formula:

$$x_{int} = h \pm \sqrt{-\frac{k}{a}} \qquad\qquad x_{int} = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}$$

If the quantity in the square root is positive, there are two solutions for x_{int}, indicating two x-intercepts. These are symmetric about the axis of the parabola. If it happens that $k = 0$, or equivalently if $B^2 - 4AC = 0$, the square root disappears, and the x-intercept is precisely at the vertex

$$x_h = h = -\frac{B}{2A} \, .$$

If the square root ultimately contains a negative number, the parabola does not touch the x-axis, and there are no x-intercepts.

Symmetry

Any parabola is mirror-symmetric about its axis, i.e. the line $x = h$. This means if we choose any location $(h - x^*, y^*)$ on the parabola, and then invert the sign on x^* such that $x^* \to -x^*$, then the height y^* of the mirror-image point $(h + x^*, y^*)$ remains the same.

8 Factoring Quadratics

8.1 Reverse Distribution

The distributive property of multiplication involves multiplying an expression *into* another expression, for instance

$$a(b+c) = ab + ac\,.$$

This operation can be 'undone' by a move called *factoring*. In general, factoring an expression means to transform a sum or difference into a product. To undo the distribution above, we would write

$$ab + ac = a(b+c)\ ,$$

and the expression is now 'factored'.

Consider next the square of a sum, $(x+y)^2$. Using the distributive property (or FOIL method), we find

$$(x+y)^2 = x^2 + 2xy + y^2\,.$$

Reading this equation in reverse, we have that whenever the form $x^2+2xy+y^2$ is encountered, the factored version must be $(x+y)^2$. Similarly, the square of a difference reads simply swaps y for $-y$ in the above:

$$x^2 - 2xy + y^2 = (x-y)^2$$

Example 1

Find the GCF and LCM of:

$$x^2y - xy^2 \qquad\qquad 3x - 3y \qquad\qquad x^2 - 2xy + y^2$$

Step 1: Factor each expression:

$$x^2 - xy^2 = xy(x-y)$$
$$3x - 2y = 3(x-y)$$
$$x^2 - 2xy + y^2 = (x-y)^2$$

Step 2: Identify the GCF:

$$GCF = (x-y)$$

Step 3: Identify the LCM:

$$LCM = 3xy(x-y)^2$$

8.2 Solutions as Zeros

Consider the quadratic equation

$$0 = Ax^2 + Bx + C\,.$$

Supposing for the moment that two solutions x_1 and x_2 exist, it follows that the same quadratic equation can be written

$$0 = A(x-x_1)(x-x_2)\ .$$

This is checked easily by setting x to either x_1 or x_2, giving zero on the right. For this reason, solutions to an equation are often called *zeros*. For any value x^* that solves the quadratic equation, the quantity $(x - x^*)$ is a factor.

If the argument seems too fast or unconvincing, recall that we already have solutions x_1, x_2 from the quadratic formula

$$x_1 = -\frac{B}{2A} + \frac{\sqrt{B^2 - 4AC}}{2A} \qquad\qquad x_2 = -\frac{B}{2A} - \frac{\sqrt{B^2 - 4AC}}{2A} \ ,$$

which can be substituted into the above

$$\begin{aligned}
0 &= A\left(x - x_1\right)\left(x - x_2\right) \\
0 &= Ax^2 - A\left(x_1 + x_2\right)x + Ax_1x_2 \\
0 &= Ax^2 - A\left(-\frac{B}{A}\right)x + A\left(\frac{B^2}{4A^2} - \frac{B^2 - 4AC}{4A^2}\right) \\
0 &= Ax^2 + Bx + C
\end{aligned}$$

to recover the original equation.

8.3 Completing the Square

It's possible to solve for x directly in a quadratic equation by a move called *completing the square*. Begin with the quadratic equation

$$0 = Ax^2 + Bx + C \ ,$$

and divide through by A to isolate the x^2-term:

$$0 = x^2 + \frac{B}{A}x + \frac{C}{A}$$

The goal is to write

$$0 = \left(x - p\right)^2 + q \ ,$$

where p, q are determined by A, B, C.

Expanding out the square term and comparing coefficients to the above, we must have

$$-2p = \frac{B}{A} \qquad\qquad p^2 + q = \frac{C}{A} \ ,$$

or

$$p = -\frac{B}{2A} \qquad\qquad q = \frac{C}{A} - \frac{B^2}{4A^2} \ .$$

Now the hard work is finished. Solve for x in the above to get

$$x = p \pm q = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \ ,$$

none other than the quadratic formula.

8.4 Special Coefficients

Factoring a quadratic expression is easier when the leading coefficient on the x^2-term is precisely one. For the case $A = 1$, the expression $x^2 + Bx + C$ factors as

$$x^2 + Bx + C = (x - x_1)(x - x_2) \ ,$$

where

$$x_1 = \frac{1}{2}\left(-B + \sqrt{B^2 - 4C}\right) \qquad\qquad x_2 = \frac{1}{2}\left(-B - \sqrt{B^2 - 4C}\right) \ .$$

Taking a moment to calculate the sum $x_1 + x_2$, we find

$$x_1 + x_2 = \frac{1}{2}\left(-B + \cancel{\sqrt{B^2 - 4C}}\right) + \frac{1}{2}\left(-B - \cancel{\sqrt{B^2 - 4C}}\right) = -B \ ,$$

and the result is simply $-B$. Calculating the product $x_1 \cdot x_2$ gives

$$x_1 \cdot x_2 = \frac{1}{4}\left(\cancel{B^2} - \cancel{B^2} + 4C\right) = C \ ,$$

which could hardly be simpler.

If the leading coefficient is not one ($A \neq 1$), we may factor A out of the expression such that

$$Ax^2 + Bx + C = A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right) = A\left(x^2 + \tilde{B}x + \tilde{C}\right) \ ,$$

effectively removing the leading coefficient, so we instead solve

$$x^2 + \tilde{B}x + \tilde{C} = (x - x_1)(x - x_2) \ ,$$

where $\tilde{B} = B/A$, and $\tilde{C} = C/A$. Similar to the above, the sum of x_1, x_2 must equal $-\tilde{B}$, and the product must equal $\tilde{C}$. While this trick is handy, it often turns out that $\tilde{B}$ and $\tilde{C}$ compute to 'ugly' numbers.

In the special case that the first and third coefficients are the same ($A = C$), we can divide through by A to get

$$Ax^2 + Bx + A = A\left(x^2 + \tilde{B}x + 1\right) \ ,$$

where $\tilde{B} = B/A$. Supposing x_1, x_2 are solutions to the equation, the factoring problem reduces to solving

$$x^2 + \tilde{B}x + 1 = 0 \ ,$$

a special application of the quadratic formula:

$$x = \frac{1}{2}\left(-\tilde{B} \pm \sqrt{\tilde{B}^2 - 4}\right)$$

Next, let us make the curious change of variables such that $z \to 1/x$, where the above becomes

$$\left(\frac{1}{z}\right)^2 + \frac{\tilde{B}}{z} + 1 = 0 \, ,$$

and multiplying through by z^2 results in the same equation we started with, with z in place of x:

$$1 + \tilde{B}z + z^2 = 0$$

Then, it must be true that if we have a solution x_1 to the first equation, then the second solution x_2 must be the reciprocal of x_1. Checking this carefully, we find:

$$x_1 = \frac{1}{2}\left(-\tilde{B} - \sqrt{\tilde{B}^2 - 4}\right)$$

$$\frac{1}{x_1} = \frac{-2}{\tilde{B} + \sqrt{\tilde{B}^2 - 4}} \cdot \frac{\tilde{B} - \sqrt{\tilde{B}^2 - 4}}{\tilde{B} - \sqrt{\tilde{B}^2 - 4}} = \frac{1}{2}\left(-\tilde{B} + \sqrt{\tilde{B}^2 - 4}\right) = x_2$$

8.5 Method of Regrouping

Now we develop a wholly general method for factoring quadratic expressions. Starting with the general form Ax^2+Bx+C, write the expression in an already-factored form, necessitating two new variables f, g such that

$$\begin{aligned} Ax^2 + Bx + C &= Ax^2 + fx + gx + C \\ &= \frac{1}{C}\left(fx + C\right)\left(gx + C\right) \, , \end{aligned}$$

where we must have

$$\begin{aligned} f + g &= B \\ f \cdot g &= AC \, . \end{aligned}$$

The above constitutes a system of two equations and two unknowns. Solving the system for f, we find a 'regrouped' equation

$$f^2 - Bf + AC = 0 \, ,$$

which is a quadratic equation with f as the variable, and the leading coefficient is *one*. We could also solve the system for g and write an identical equation, which is in fact not necessary, as the f-equation alone produces both solutions:

$$f = \frac{1}{2}\left(B + \sqrt{B^2 - 4AC}\right) \qquad\qquad g = \frac{1}{2}\left(B - \sqrt{B^2 - 4AC}\right)$$

Of course, the idea is to *avoid* calculating anything like $\sqrt{B^2 - 4AC}$, as it's often easier to solve $f^2-Bf+AC = 0$ 'by eye'. In the special case $B = 0$, we readily have $g = -f = \sqrt{-AC}$.

Example 2

Factor:

$$6x^2 + 11x + 4$$

Step 1: Identify coefficients:

$$A = 6 \qquad B = 11 \qquad C = 4 \qquad AC = 24$$

Step 2: Write the transformed equation:

$$f^2 - 11f + 24 = 0$$

Step 3: Solve the transformed equation:

$$(f - 8)(f - 3) = 0$$

Step 4: Identify $f = 8$ and $g = 3$, and rewrite the original expression:

$$6x^2 + 11x + 4 = \frac{1}{4}(8x + 4)(3x + 4)$$

Step 5: Simplify to get the result:

$$6x^2 + 11x + 4 = (2x + 1)(3x + 4)$$

Example 3

Factor:

$$15x^2 + 14x - 8$$

Step 1: Identify coefficients:

$$A = 15 \qquad B = 14 \qquad C = -8 \qquad AC = -120$$

Step 2: Write the transformed equation:

$$f^2 - 14f - 120 = 0$$

Step 3: Solve the transformed equation:

$$(f - 20)(f + 6) = 0$$

Step 4: Identify $f = 20$ and $g = -6$, and rewrite the original expression:

$$15x^2 + 14x - 8 = -\frac{1}{8}(20x - 8)(-6x - 8)$$

Step 5: Simplify to get the result:

$$\begin{aligned} 15x^2 + 14x - 8 &= \frac{(20x - 8)}{4}\frac{(6x + 8)}{2} \\ &= (5x - 2)(3x + 4) \end{aligned}$$

Example 4

Factor:

$$6x^2 - 4x - 16$$

Step 1: Identify coefficients:

$$A = 6 \qquad B = -4 \qquad C = -16 \qquad AC = -96$$

Step 2: Write the transformed equation:

$$f^2 + 4f - 96 = 0$$

Step 3: Solve the transformed equation:

$$(f + 12)(f - 8) = 0$$

Step 4: Identify $f = -12$ and $g = 8$, and rewrite the original expression:

$$6x^2 - 4x + 16 = -\frac{1}{16}(-12x - 16)(8x - 16)$$

Step 5: Simplify to get the result:

$$\begin{aligned} 6x^2 - 4x - 16 &= \frac{(12x + 16)}{2}\frac{(8x - 16)}{8} \\ &= (6x + 8)(x - 2) \end{aligned}$$

<u>Example 5</u>

Factor:

$$-2x^2 - 6x + 56$$

Step 1: Identify coefficients:

$$A = -2 \qquad B = -6 \qquad C = 56 \qquad AC = -112$$

Step 2: Write the transformed equation:

$$f^2 - 6f - 112 = 0$$

Step 3: Solve the transformed equation:

$$(f + 14)(f - 8) = 0$$

Step 4: Identify $f = -14$ and $g = 8$, and rewrite the original expression:

$$-2x^2 - 6x + 56 = \frac{1}{56}(-14x + 56)(8x + 56)$$

Step 5: Simplify to get the result:

$$\begin{aligned} -2x^2 - 6x + 56 &= \frac{(-14x + 56)}{7}\frac{(8x + 56)}{8} \\ &= (-2x + 8)(x + 7) \\ &= -2(x - 4)(x + 7) \end{aligned}$$

Example 6

Factor:

$$24x^2 - 6xy - 9y^2$$

Step 1: Factor a 3 out of the expression:

$$24x^2 - 6xy - 9y^2 = 3\left(8x^2 - 2xy - 3y^2\right)$$

Step 2: Identify coefficients:

$$A = 8 \qquad B = -2y \qquad C = -3y^2 \qquad AC = -24y^2$$

Step 3: Write the transformed equation:

$$f^2 + 2yf - 24y^2 = 0$$

Step 4: Solve the transformed equation:

$$(f + 6y)(f - 4y) = 0$$

Step 5: Identify $f = -6y$ and $g = 4y$, and rewrite the original expression:

$$\begin{aligned} 24x^2 - 6xy - 9y^2 &= 3\left(8x^2 - 2xy - 3y^2\right) \\ &= 3 \times \frac{1}{-3y^2}\left(-6xy - 3y^2\right)\left(4xy - 3y^2\right) \end{aligned}$$

Step 6: Simplify to get the result:

$$\begin{aligned} 24x^2 - 6xy - 9y^2 &= -\frac{\left(-6xy - 3y^2\right)}{y}\frac{\left(4xy - 3y^2\right)}{y} \\ &= 3(2x + y)(4x - 3y) \end{aligned}$$

Example 7

Factor:

$$36x^2 - 121y^2$$

Step 1: Identify coefficients:

$$A = 36 \qquad B = 0 \qquad C = -121y^2$$

Step 2: Write the transformed equation:

$$f^2 - 36 \cdot 121y^2 = 0$$

Step 3: Solve the transformed equation:

$$f = \pm 6 \cdot 11y = \pm 66y$$

Step 4: Identify $f = 66y$ and $g = -66y$, and rewrite the original expression:

$$36x^2 - 121y^2 = -\frac{1}{121y^2}\left(66xy - 121y^2\right)\left(-66xy - 121y^2\right)$$

Step 5: Simplify to get the result:

$$\begin{aligned} 36x^2 - 121y^2 &= \frac{(66x - 121y)}{11}\frac{(-66x - 121y)}{11} \\ &= (6x - 11y)(6x + 11y) \end{aligned}$$

8.6 Applications

Nonlinear System

Consider the following nonlinear system of two equations and two unknowns:

$$2x^2 + 3y = 19$$
$$4x - y = 3$$

To solve for x and y analytically, we first need to combine the two equations to eliminate one of the variables. Multiplying the second equation through by a factor of three, and then adding the result to the first equation, we find

$$x^2 + 6x = 14\ ,$$

which is clearly a quadratic equation, having two solutions:

$$x = -3 \pm \frac{1}{2}\sqrt{6^2 + 4 \cdot 14} \qquad \rightarrow \qquad \begin{cases} x_1 = -3 + \sqrt{23} \approx 1.796 \\ x_2 = -3 - \sqrt{23} \approx -7.796 \end{cases}$$

Now, the 'usual' technique is to substitute the solution for x into either equation, and then solve for y, and we're done. However, there are *two* x-values on hand, so which do we use? This question is amplified when we try the exercise of solving for y, eliminating x. Carrying this out, we find another quadratic equation

$$y^2 + 30y = 143\ ,$$

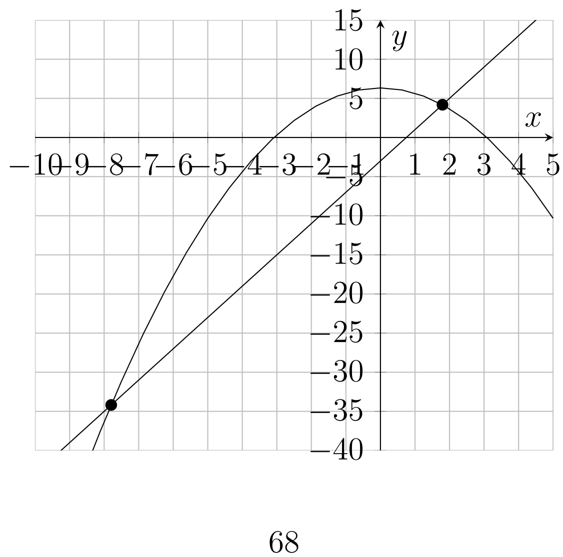
having two solutions:

$$y = -15 \pm \frac{1}{2}\sqrt{30^2 + 4 \cdot 143} \qquad \rightarrow \qquad \begin{cases} y_1 = -15 + \sqrt{368} \approx 4.183 \\ y_2 = -15 - \sqrt{368} \approx -34.183 \end{cases}$$

At this stage, we have four possible solutions to the system

$$(x_1, y_1) \qquad (x_1, y_2) \qquad (x_2, y_1) \qquad (x_2, y_2)\ ,$$

however not all be valid. Each can easily be checked against the original equations, which is equivalent to checking with a graphical method. Solving each equation for y and plotting each on the same graph, we see the intersection of a parabola and a line:



Evidently, intersections occur at $(1.796, 4.183)$ and $(-7.796, -34.183)$, thus the valid solutions to the system are (x_1, y_1) and (x_2, y_2).

9 Cubic Equations

As a generalization to the quadratic expression, a *cubic expression* includes a third-order term multiplied by a coefficient:

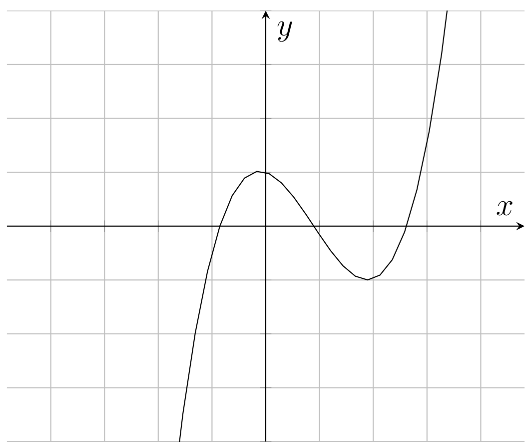
$$Ax^3 + Bx^2 + Cx + D$$

9.1 Features of Cubic Equations

The formula

$$y = Ax^3 + Bx^2 + Cx + D$$

constitutes a *cubic equation*, generally appearing as follows:



In the above, the coefficients are chosen such that

$$A = \frac{1}{2} \qquad B = -\frac{4}{3} \qquad C = -\frac{1}{3} \qquad D = 1\,.$$

Coefficients

The leading coefficient A determines the overall 'climb' of the plot. For $A > 0$, the plot grows upward for increasing x, while dipping (very) negative for decreasing x. This is true regardless of B, C, D, as the x^3-term grows much faster than the lower-order terms. For $A < 0$, the plot grows downward as x increases, whereas largely-negative x-values lead to very large y-values.

The coefficients B and C are responsible for the overall structure of the plot near the origin (for not-so-large values of x). The coefficient D plays the role of the y-intercept, controlling the vertical placement of the plot on the Cartesian graph.

x-Intercept(s)

In the general case, a cubic equation flaunts two vertex points and up to three x-intercepts. Depending on the values of B, C, D, the number of x-intercepts can vary, but there are *never* zero.

9.2 Solving Cubic Equations

We now develop a method to solve cubic equations, which amounts to looking for x-intercepts in the cubic equation

$$0 = Ax^3 + Bx^2 + Cx + D\,.$$

Depressed Cubic

To begin, we effectively do away with the x^2-term by making the substitution

$$x = z - \frac{B}{3A}$$

as follows:

$$\begin{aligned}
0 &= A\left(z - \frac{B}{3A}\right)^3 + B\left(z - \frac{B}{3A}\right)^2 + C\left(z - \frac{B}{3A}\right) + D \\
0 &= A\left(z^3 - \cancel{3z^2\frac{B}{3A}} + 3z\left(\frac{B}{3A}\right)^2 - \left(\frac{B}{3A}\right)^3\right) \\
&\quad + B\left(\cancel{z^2} - 2z\frac{B}{3A} + \left(\frac{B}{3A}\right)^2\right) + C\left(z - \frac{B}{3A}\right) + D \\
0 &= Az^3 + z\left(-\frac{B^2}{3A} + C\right) - A\left(\frac{B}{3A}\right)^3 + B\left(\frac{B}{3A}\right)^2 - C\left(\frac{B}{3A}\right) + D \\
0 &= z^3 + z\left(-\frac{B^2}{3A^2} + \frac{C}{A}\right) + 2\left(\frac{B}{3A}\right)^3 - \frac{C}{A}\left(\frac{B}{3A}\right) + \frac{D}{A}
\end{aligned}$$

Proceed by setting

$$a = -\frac{B^2}{3A^2} + \frac{C}{A} \qquad\qquad -b = 2\left(\frac{B}{3A}\right)^3 - \frac{C}{A}\left(\frac{B}{3A}\right) + \frac{D}{A}$$

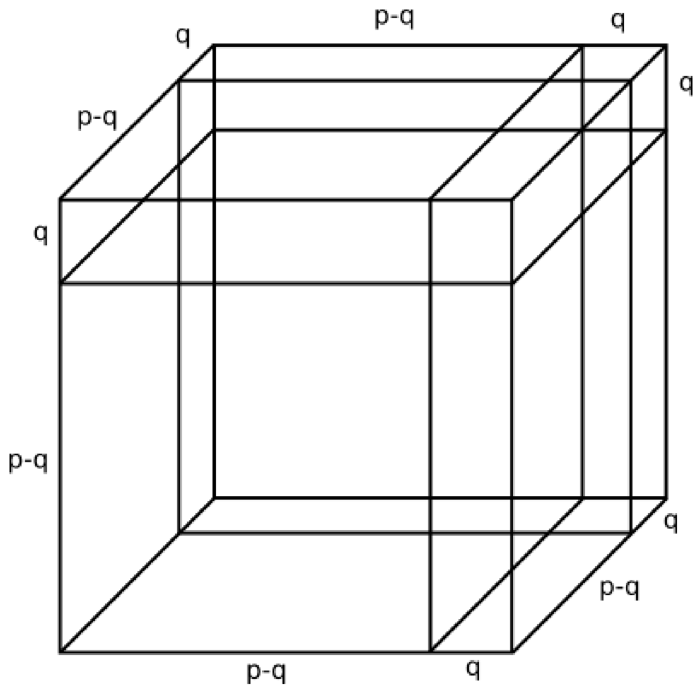
to arrive at the equation of the *depressed cubic*:

$$z^3 + az = b$$

Since we're operating in a general scheme, it's clear that any cubic equation can be 'depressed' into the form above, and then the problem becomes finding solutions for z in terms of a and b. With this understood, let us make switch of notation $z \rightarrow x$ for convenience, knowing the true x can be restored by adding $B/3A$.

Geometric Interpretation

The depressed cubic equation can be solved by a trick attributed to Gerolamo Cardano in 1545. Tracing Cardano's steps, begin with a cube of side p, and then introduce three planes inside the cube, parallel to the top, right, and back faces. Set each plane length q from the cube's respective sides as shown.



The total volume illustrated consists of the 'main' cube of side $p-q$, three slabs of volume $q\left(p-q\right)^2$, three bars of volume $q^2\left(p-q\right)$, and a small cube of side q. Meanwhile, the total volume is simply p^3, allowing us to write

$$p^3 = \left(p-q\right)^3 + 3q^2\left(p-q\right) + 3\left(p-q\right)^2 q + q^3 \; ,$$

readily simplifying to

$$\left(p-q\right)^3 + 3pq\left(p-q\right) = p^3 - q^3 \; ,$$

which is indeed another equation for the depressed cubic equation. Identifying

$$p - q = x \qquad\qquad 3pq = a \qquad\qquad p^3 - q^3 = b \; ,$$

we recover the form $x^3 + ax = b$.

Eliminating q between the latter two equations, we end up with

$$p^6 - bp^3 - \left(\frac{a}{3}\right)^3 = 0 \; ,$$

which is in fact a quadratic equation in the variable p^3, easily isolated by the quadratic formula:

$$p^3 = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}$$

This automatically gives us q^3, specifically

$$q^3 = p^3 - b = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3} \; ,$$

allowing a solution for x to be written:

$$x = \left(\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}\right)^{1/3} - \left(-\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}\right)^{1/3}$$

Recall though we are working in the shifted variable $x \to x - B/3A$, so the factor $B/3A$ must be added to the above to account for this.

<u>Example 1</u>

Find one solution to:

$$x^3 - \frac{x}{3} - \frac{2}{27} = 0$$

Step 1: Identify the above as a depressed cubic equation and pick out coefficients:

$$3pq = a = -\frac{1}{3} \qquad\qquad p^3 - q^3 = b = \frac{2}{27}$$

Step 2: Solve for p^3, and write p and q:

$$p^3 = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3} = \frac{1}{27} \pm \sqrt{\left(\frac{1}{27}\right)^2 - \left(\frac{1}{9}\right)^3} = \frac{1}{27}$$

$$p = \frac{1}{3} \qquad\qquad q = -\frac{1}{3}$$

Step 3: Write the solution for x in terms of p and q:

$$x = p - q = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Two More Solutions

In the general case, the geometric approach to solving a depressed cubic equation

$$x^3 + ax = b$$

produces a solution, however there should exist (up to) three total solutions to the equation. Labeling the known solution as w, it follows that $(x - w)$ can be factored out of the depressed cubic equation according to

$$x^3 + ax - b = (x - w)\left(x^2 + tx + u\right) \ ,$$

where coefficients t, u are determined by a, b. By comparing same-order terms in x, we find

$$t = w \qquad\qquad u = \frac{b}{w}$$

to get

$$x^3 + ax - b = (x - w)\left(x^2 + wx + \frac{b}{w}\right) \ .$$

Remaining solutions to the depressed cubic equation are given by the zeros of the expression

$$x^2 + wx + \frac{b}{w}\,,$$

an application of the quadratic formula:

$$x = \frac{w}{2}\left(-1 \pm \sqrt{1 - \frac{4b}{w^3}}\right)$$

Example 2

Use the known solution $w = 2/3$ to continue factoring the expression:

$$x^3 - \frac{x}{3} - \frac{2}{27}$$

Step 1: Substitute $w = 2/3$ and $b = -2/27$ into the equation for x and simplify:

$$x = \frac{2/3}{2}\left(-1 \pm \sqrt{1 - \frac{4 \cdot 2/27}{(2/3)^3}}\right) = -\frac{1}{3}$$

Step 2: Pick out any new solution(s) gained. In this case, we have two copies of the same solution:

$$x_1 = x_2 = -\frac{1}{3}$$

Step 3: Write the final form:

$$x^3 - \frac{x}{3} - \frac{2}{27} = \left(x - \frac{2}{3}\right)\left(x + \frac{1}{3}\right)^2$$

9.3 Applications

Depressed Cubic in Disguise

Consider the rather exotic quantity

$$\left(7 + \sqrt{50}\right)^{1/3} + \left(7 - \sqrt{50}\right)^{1/3}\,,$$

which might seem impossible to evaluate, namely because $7 - \sqrt{50}$ is surely negative. Proceeding boldly, let us store the whole quantity in a variable x, and then calculate x^3:

$$\begin{aligned}
x^3 &= \left(\left(7 + \sqrt{50}\right)^{1/3} + \left(7 - \sqrt{50}\right)^{1/3}\right)^3 \\
&= 7 + \sqrt{50} + 3\left(7 + \sqrt{50}\right)^{2/3}\left(7 - \sqrt{50}\right)^{1/3} + 3\left(7 + \sqrt{50}\right)^{1/3}\left(7 - \sqrt{50}\right)^{2/3} + 7 - \sqrt{50} \\
&= 14 + 3\left(\left(7 + \sqrt{50}\right)^{1/3}\left(7 - \sqrt{50}\right)^{1/3}\right)\left(\left(7 + \sqrt{50}\right)^{1/3} + \left(7 - \sqrt{50}\right)^{1/3}\right) \\
&= 14 + 3\,(-1)^{1/3}\,x
\end{aligned}$$

Evidently then, we find

$$x^3 = -3x + 14\,,$$

the equation of a depressed cubic. From here, it's probably easier to guess that $x = 2$ rather than deploy the rest of the machinery, however. With the value for x found, we write the answer:

$$\left(7 + \sqrt{50}\right)^{1/3} + \left(7 - \sqrt{50}\right)^{1/3} = 2$$

10 Polynomial Division

A brute-force method for manipulating an expression entails long division of polynomials. The setup and procedure for polynomial division is identical to elementary methods for arithmetic. To illustrate polynomial division, consider a ratio such as

$$\frac{x^2 - 9x - 10}{x + 1},$$

and set up the corresponding division problem as follows:

$$
\begin{array}{r|rrr}
\cline{2-4}
x+1 & x^2 & -9x & -10
\end{array}
$$

To proceed, divide the first term in the dividend by the first term in the divisor to get $x^2/x = x$. Place the result (x) in the quotient field. Then, distribute x into the divisor and subtract the result from the dividend:

$$
\begin{array}{r|rrr}
 & & x & \\
\cline{2-4}
x+1 & x^2 & -9x & -10 \\
 & x^2 & x & \\
\cline{2-4}
 & & -10x & -10
\end{array}
$$

We may now regard $-10x - 10$ as the updated dividend and repeat the process. Dividing the respective leading terms, we find $-10x/x = -10$, and update our progress as follows:

$$
\begin{array}{r|rrr}
 & & x & -10 \\
\cline{2-4}
x+1 & x^2 & -9x & -10 \\
 & x^2 & x & \\
\cline{2-4}
 & & -10x & -10 \\
 & & -10x & -10 \\
\cline{3-4}
 & & & 0
\end{array}
$$

With a new dividend of zero, the process halts, and we can read off the answer:

$$\frac{x^2 - 9x - 10}{x + 1} = x - 10$$

10.1 Remainders

Taking a more informative example, consider the ratio

$$\frac{\left(x^4 + x + 1\right)^2}{x^2 - 1} = \frac{x^8 + 2x^5 + 2x^4 + x^2 + 2x + 1}{x^2 - 1}.$$

Note that the numerator should always contain a higher-degree polynomial than the denominator. Setting up and starting the hard work, we have:

$$
\begin{array}{r|rrrrrrrr}
 & & x^6 & & +x^4 & +2x^3 & +3x^2 & +2x & +4 \\ \hline
x^2-1 & x^8 & & +2x^5 & +2x^4 & & +x^2 & +2x & +1 \\
 & x^8 & -x^6 & & & & & & \\ \cline{2-9}
 & & x^6 & +2x^5 & +2x^4 & & +x^2 & +2x & +1 \\
 & & x^6 & & -x^4 & & & & \\ \cline{3-9}
 & & & 2x^5 & +3x^4 & & +x^2 & +2x & +1 \\
 & & & 2x^5 & & -2x^3 & & & \\ \cline{4-9}
 & & & & 3x^4 & +2x^3 & +x^2 & +2x & +1 \\
 & & & & 3x^4 & & -3x^2 & & \\ \cline{5-9}
 & & & & & 2x^3 & +4x^2 & +2x & +1 \\
 & & & & & 2x^3 & & -2x & \\ \cline{6-9}
 & & & & & & 4x^2 & +4x & +1 \\
 & & & & & & 4x^2 & & -4 \\ \cline{7-9}
 & & & & & & & 4x & +5
\end{array}
$$

The next step *would* be to try dividing $4x$ by x^2, however the result (and any following it) will contain factors of x^{-1}. Reading off the quotient while regarding $(4x+5)\,/\,(x^2-1)$ as the remainder, we assemble the result:

$$
\frac{x^8 + 2x^5 + 2x^4 + x^2 + 2x + 1}{x^2 - 1} = x^6 + x^4 + 2x^3 + 3x^2 + 2x + 4 + \frac{4x+5}{x^2-1}
$$

10.2 Application to Factoring

Suppose we are interested in factoring a quantity $x^n - a$, where n is any integer greater than zero. In general, the number of zeros, i.e. number of potential solutions to

$$
x^n - a = 0 \, ,
$$

is equal to n, the highest-degree term in the expression. At the very least though, we can be sure that $x = a^{1/n}$ is one valid solution.

Using polynomial division, a term $x - a^{1/n}$ can be factored out of $x^n - a$. Setting up the division problem, we write:

$$
\begin{array}{r|rr}
 & & \\ \hline
x - a^{1/n} & x^n & -a
\end{array}
$$

Without specifying n, it's not clear where the division process ought to terminate. Carrying out the division steps *anyway*, we find, after four steps:

$$
\frac{x^n - a}{x - a^{1/n}} = x^{n-1} + a^{1/n}x^{n-2} + a^{2/n}x^{n-3} + a^{3/n}x^{n-4} + \frac{a^{4/n}x^{n-4} - a}{x - a^{1/n}}
$$

Evidently, the maximum number of division steps should not exceed the degree number n, otherwise the exponent on x becomes negative. Tidy up the equation by multiplying $x - a^{1/n}$

into each side:

$$x^n - a = \left(x - a^{1/n}\right)\left(x^{n-1} + a^{1/n}x^{n-2} + a^{2/n}x^{n-3} + a^{3/n}x^{n-4}\right) + \left(a^{4/n}x^{n-4} - a\right)$$

Next, suppose instead of strictly four steps, the division process carries out j steps, where we now have

$$x^n - a = \left(x - a^{1/n}\right)\left(x^{n-1} + a^{1/n}x^{n-2} + a^{2/n}x^{n-3} + \cdots + a^{(j-1)/n}x^{n-j}\right) + \left(a^{j/n}x^{n-j} - a\right) .$$

If j is tuned to equal n, the remainder term vanishes, and the above becomes

$$x^n - a = \left(x - a^{1/n}\right)\left(x^{n-1} + a^{1/n}x^{n-2} + a^{2/n}x^{n-3} + \cdots + a^{(n-2)/n}x + a^{(n-1)/n}\right) .$$

Summation Notation

In order to avoid writing a long polynomial battered with exponents, we can 'spot the pattern' in the sum, and use condensed notation as follows:

$$x^n - a = \left(x - a^{1/n}\right)\left(\sum_{k=1}^{n} a^{(k-1)/n}x^{n-k}\right)$$

This result achieves the task of factoring a known solution $x - a^{1/n}$ out of $x^n - a$. The price we pay is that all other solutions are embedded in a potentially long polynomial.

Example 1

Factor:

$$x^3 - 8$$

Step 1: Identify variables:

$$n = 3 \qquad\qquad a = 8$$

Step 2: Write the factored expression in summation notation:

$$x^3 - 8 = \left(x - 8^{1/3}\right)\left(\sum_{k=1}^{3} 8^{(k-1)/3}x^{3-k}\right)$$

Step 3: Simplify:

$$x^3 - 8 = (x-2)\left(x^2 + 2x + 4\right)$$

Example 2

Factor:

$$x^4 - 9$$

Step 1: Identify variables:

$$n = 4 \qquad\qquad a = 9$$

Step 2: Write the factored expression in summation notation:

$$x^4 - 9 = \left(x - 9^{1/4}\right)\left(\sum_{k=1}^{4} 9^{(k-1)/4} x^{4-k}\right)$$

Step 3: Simplify:

$$\begin{aligned} x^4 - 9 &= \left(x - \sqrt{3}\right)\left(x^3 + \sqrt{3}x^2 + 3x + 3\sqrt{3}\right) \\ &= \left(x - \sqrt{3}\right)\left(x^2\left(x + \sqrt{3}\right) + 3\left(x + \sqrt{3}\right)\right) \\ &= \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)\left(x^2 + 3\right) \end{aligned}$$

11 Partial Fractions

While polynomial division is best-suited for breaking apart 'top-heavy' ratios, another technique is needed to grapple with 'bottom-heavy' ratios, called *partial fractions*. Starting with the case where the denominator has a degree-two polynomial in factored form, observe that such a ratio can be can be split into the sum of two terms, each containing a degree-one polynomial:

$$\frac{cx+d}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

The unknowns A, B are easily determined in terms of a, b, c d. By setting $x = 0$, and then $x = 1/c$, respectively, we gain two equations

$$bA + aB = -d \qquad\qquad c = A + B\,,$$

solved by

$$A = \frac{ac+d}{a-b} \qquad\qquad B = \frac{bc+d}{b-a}\,,$$

which could have been inferred by choosing values $x = a$, $x = b$. This method generalizes to higher-degree polynomial denominators, as shown for the degree-three case:

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

Example 1

Find the equivalent ratio as a sum of partial fractions:

$$\frac{2x+1}{(x-3)(x-4)}$$

Step 1: Rewrite the ratio as a sum:

$$\frac{2x+1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

Step 2: Solve for A and B to get:

$$A = -7 \qquad\qquad B = 9$$

Step 3: Assemble the result:

$$\frac{2x+1}{(x-3)(x-4)} = \frac{-7}{x-3} + \frac{9}{x-4}$$

Example 2

Find the equivalent ratio as a sum of partial fractions:

$$\frac{1}{a^2-x^2}$$

Step 1: Factor the denominator:

$$\frac{1}{a^2 - x^2} = \frac{1}{(a-x)(a+x)}$$

Step 2: Rewrite the ratio as a sum:

$$\frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x}$$

Step 3: Solve for A and B to get:

$$A = \frac{1}{2a} \qquad\qquad B = \frac{1}{2a}$$

Step 4: Assemble the result:

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left(\frac{1}{a-x} + \frac{1}{a+x}\right)$$

11.1 Repeated Roots

Of course, the partial fraction expansion is prone to error if we run into division by zero, i.e. the case $a = b$. To handle a ratio having two repeated roots in the denominator, we use a partial fraction expansion

$$\frac{1}{(x-a)^2(x-b)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{B}{x-b} \; ,$$

admitting a separate term for each instance of $(x - a)$. This pattern generalizes to three repeated roots, and so on:

$$\frac{1}{(x-a)^3(x-b)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \frac{B}{x-b}$$

11.2 Quadratic Factors

Factors of the form $x^2 + ax + b$ occurring in the denominator can be balanced by an $Ax + B$-term according to

$$\frac{1}{(x^2+ax+b)(x-c)} = \frac{Ax+B}{x^2+bx+c} + \frac{C}{x-c} \; .$$

If a factor like $(x^2 + ax + b)^2$ occurs, extra terms are needed:

$$\frac{1}{(x^2+ax+b)^2(x-c)} = \frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \frac{C}{x-c}$$

<u>Example 3</u>

Find the equivalent ratio as a sum of partial fractions:

$$\frac{1}{x^4 - 1}$$

Step 1: Factor the denominator:

$$\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)}$$

Step 2: Rewrite the ratio as a sum:

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

Step 3: Multiply through by $(x-1)(x+1)(x^2+1)$:

$$1 = A(x+1)\left(x^2+1\right) + B(x-1)\left(x^2+1\right) + (Cx+D)(x-1)(x+1)$$

Step 4: Let $x = 1$, $x = -1$, $x = 0$, and $x = 2$ to isolate each coefficient:

$$A = \frac{1}{4} \qquad B = -\frac{1}{4} \qquad D = -\frac{1}{2} \qquad C = 0$$

Step 5: Assemble the result:

$$\frac{1}{x^4-1} = \frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) - \frac{1}{2}\left(\frac{1}{x^2+1}\right)$$

11.3 Mixed Cases

Certain situations call for polynomial division *and* partial fractions. For instance, in the ratio

$$\frac{x^3+4}{x^2+x},$$

the numerator contains a higher-degree polynomial than the denominator. Carrying out the division problem

$$x^2+x \overline{\big)\; x^3 \quad +4 \quad},$$

we end up with a quotient and a remainder as follows:

$$\frac{x^3+4}{x^2+x} = (x-1) + \frac{x+4}{x^2+x}$$

Next, take the remainder term in isolation and use partial fraction analysis to write

$$\frac{x+4}{x^2+x} = \frac{x+4}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1},$$

where we easily get $A = 4$, $B = -3$. In summary then, we find:

$$\frac{x^3+4}{x^2+x} = x - 1 + \frac{4}{x} - \frac{3}{x+1}$$

12 Geometric Series

Let us pay special attention to the ratio

$$\frac{x^n - 1}{x - 1} \,,$$

which is the $a = 1$-case of the previously-established equation

$$x^n - a = \left(x - a^{1/n}\right) \left(\sum_{k=1}^{n} a^{(k-1)/n} x^{n-k} \right)$$

derived by polynomial division. Choosing a few values for n, we may write

$$\begin{aligned} \frac{x^2 - 1}{x - 1} &= 1 + x \\ \frac{x^3 - 1}{x - 1} &= 1 + x + x^2 \\ \frac{x^4 - 1}{x - 1} &= 1 + x + x^2 + x^3 \,, \end{aligned}$$

where for a general n, the above becomes

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1} = \sum_{k=1}^{n} x^{k-1} \,.$$

Before proceeding, let us break apart the left side of the equation to move all n-dependence to the right:

$$\frac{1}{1 - x} = \sum_{k=1}^{n} x^{k-1} + \frac{x^n}{1 - x}$$

Note that the number of zeros in $(x^n - 1) / (x - 1)$ is equal to n, the degree of the numerator. Meanwhile, as n increases, the number of polynomial terms on the right side of the equations grows steadily and predictably. If we take n to be *infinitely* large, the number of zeros approaches infinity, and so does the number of terms in the polynomial on the right. This is a rather boring scenario if x^n itself approaches infinity, however if x satisfies $-1 < x < 1$, then x^n approaches *zero*. With n set to infinity, the above becomes

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots \qquad\qquad -1 < x < 1 \,,$$

called the *geometric series*. The geometric series applies to the domain $-1 < x < 1$, called the *basin of convergence*.

12.1 Alternate Derivations

The geometric series can be derived in several ways. While we have taken a semi-formal approach, some shortcuts can help recover the series in a pinch. If you know the left side is the ratio $1 / (1 - x)$, the right side an be derived from long division to get

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1 - x} \,.$$

If you can remember the 'series' part only, let the series equal any variable such as G, then multiply through by x to write xG:

$$\begin{aligned} G &= 1 + x + x^2 + x^3 + \cdots + x^n \\ xG &= x + x^2 + x^3 + x^4 + \cdots + x^{n+1} \end{aligned}$$

Next, take the difference $G - xG$

$$G - xG = 1 + x - x + x^2 - x^2 + x^3 - x^3 + \cdots - x^{n+1} \, ,$$

and observe that most terms cancel. Factor $(1 - x)$ from the left side to get

$$G\,(1 - x) = 1 - x^{n+1} \, ,$$

and finally:

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

<u>Example 1</u>

A basketball is dropped from 10 feet and bounces up 6 feet. On each bounce, the ball recovers 3/5 of its previous height. What is the total distance traveled by the ball, supposing it bounces forever?

Step 1: Add up the total distance accumulated during each movement downward:

$$\begin{aligned} D_1 &= 10 + \frac{3}{5} \cdot 10 + \left(\frac{3}{5}\right)^2 \cdot 10 + \left(\frac{3}{5}\right)^3 + \cdots \\ D_1 &= 10 \cdot \left(1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots\right) \end{aligned}$$

Step 2: Add up the total distance accumulated during each movement upward:

$$\begin{aligned} D_2 &= 6 + \frac{3}{5} \cdot 6 + \left(\frac{3}{5}\right)^2 \cdot 6 + \left(\frac{3}{5}\right)^3 + \cdots \\ D_2 &= 6 \cdot \left(1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots\right) \end{aligned}$$

Step 3: Compare each infinite sequence to the geometric series, and note that:

$$1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots = \frac{1}{1 - 3/5} = \frac{5}{2}$$

Step 4: Assemble the total distance moved in feet:

$$D_1 + D_2 = 10 \cdot \frac{5}{2} + 6 \cdot \frac{5}{2} = 40$$

12.2 Repeating Decimals

The geometric series can be used to make sense of decimal numbers, particularly numbers with repeating decimals. Consider any decimal number N with any number of repeating digits:

$$N = 0.abcd\ldots abcd\ldots .$$

If the number of repeating digits is Q, an equivalent representation of N is given by

$$N = \frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots + \frac{a/10}{10^Q} + \frac{b/100}{10^Q} + \frac{c/1000}{10^Q} + \cdots$$
$$N = \left(\frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots\right)\left(1 + \frac{1}{10^Q} + \frac{1}{10^{2Q}} + \frac{1}{10^{3Q}} + \cdots\right),$$

where the sequence on the right is none other than the geometric series with $x = 1/\left(10^Q\right)$. Simplifying, we have

$$N = \left(\frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots\right)\left(\frac{1}{1 - 10^{-Q}}\right),$$

where the sequence on the left is *non-repeating*, i.e., terminates after Q terms. If the last unique digit is q, we write

$$N = (0.abcd\ldots q)\left(\frac{1}{1 - 10^{-Q}}\right).$$

Multiplying the numerator and denominator by 10^Q, we can shed the decimal notation and treat $abcd\ldots q$ as a large integer such that

$$N = \frac{abcd\ldots q}{10^Q - 1}.$$

Evidently, any repeating decimal can be written in the fractional form above. It's also clear why non-repeating decimals cannot be represented as a fraction, as an infinite number of digits $abcd\ldots$ would be needed, and 10^Q also becomes infinite.

12.3 Squaring the Series

It's possible to derive an infinite series expansion for $1/\left(1 - x\right)^2$ by squaring the geometric series. Carrying this out carefully, we find

$$\begin{aligned}\left(1 + x + x^2 + x^3 + \cdots\right)^2 = 1 &+ x + x^2 + x^3 + \cdots \\ &+ x + x^2 + x^3 + x^4 + \cdots \\ &+ x^2 + x^3 + x^4 + x^5 \cdots ,\end{aligned}$$

simplifying to

$$\frac{1}{\left(1 - x\right)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

In the same spirit, the geometric series can be raied to higher powers. Listing the first few, we have:

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \cdots$$
$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \cdots$$

12.4 Change of Variables

Interesting results can be attained with a change of variables. Starting with a standard geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ,$$

we can let $x \to -x$ to discover an alternating sequence

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots .$$

This has similar consequence in the 'squared' version, where

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

becomes

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots .$$

The same goes for higher powers:

$$\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots$$
$$\frac{1}{(1+x)^4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4 + \cdots$$

Instead letting $x \to x^2$, we get a result having only even terms

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots ,$$

and similarly,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots .$$

As a sanity check, we can verify that $1/\left(1-x\right)+1/\left(1+x\right)$ produces the proper series, which indeed checks out:

$$\frac{1}{1-x} + \frac{1}{1+x} = \left(1 + x + x^2 + x^3 + \cdots\right) + \left(1 - x + x^2 - x^3 - \cdots\right)$$
$$\frac{2}{1-x^2} = 1 + 1 + x - x + x^2 + x^2 + x^3 - x^3 + \cdots$$
$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots$$

If we take $z = 1 - x$, having basin of convergence $0 < z < 2$, we write

$$\frac{1}{1-x} = \frac{1}{z} = 1 + (1-z) + (1-z)^2 + (1-z)^3 + \cdots .$$

Now, observe that each power of $(1-z)$ occurs on the right side, which means the series is equivalent to the full 'negative version' of the Pascal triangle. Plucking off (literally removing) the right-most coefficient from each row of Pascal's triangle, and assigning climbing powers of z, we have a sequence

$$1 - z + z^2 - z^3 + z^4 - \cdots .$$

With the modified triangle, remove the 'new' right-most coefficient from each row, and line up these terms to get

$$1 - 2z + 3z^2 - 4z^3 + \cdots .$$

This pattern can be repeated indefinitely, giving rise to the next sequence

$$1 - 3z + 6z^2 - 10z^3 + \cdots .$$

and so on. Note however that these sequences are not news. Indeed, each is a geometric series, respectively equal to

$$\frac{1}{1+z} \qquad \frac{1}{(1+z)^2} \qquad \frac{1}{(1+z)^3} ,$$

which establish a clear pattern. Evidently then, we find:

$$\frac{1}{z} = \frac{1}{1+z} + \frac{1}{(1+z)^2} + \frac{1}{(1+z)^3} + \cdots$$

If we change variables once more, namely $y = 1/(1+z)$, the original geometric series pops back out:

$$\frac{1}{z} = \frac{1}{1-y} = 1 + y^2 + y^3 + \cdots$$

12.5 Zeno's Paradox

An ancient 'paradox' originating in Greece began with Zeno of Elia, as recalled by Aristotle:

That which is in locomotion must arrive at the half-way stage before it arrives at the goal.

This sounds fine, but then the ancient Greeks take the argument off the rails:

In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.

In other words, according to Zeno, a moving object can never reach its destination, as it must go half-way *first*, but to reach the half-way point, it has to reach the quarter-way point, and so on. The object will thus never reach its destination, and even worse, it's not clear where it gets stuck, or if the motion ever started at all.

Spatial Sum

While Zeno's paradox has continued to keep philosophers busy over the centuries, it is of little concern to anyone aware of geometric series. Consider a racetrack that has a length of precisely one unit, and at the start of this track we place a frog or other jumping animal. Following a program of jumps inspired by Zeno, suppose the frog jumps toward the end of the track one half-unit, and then one quarter-unit, and the one eight-unit, and so on. Adding up the length of each jump, the frog moves total distance

$$D = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots .$$

Add one to each side to quickly pick out a geometric series with $x = 1/2$:

$$D + 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - 1/2} = 2 \, ,$$

or $D = 1$.

We find that jumping 'halfway there' at every step is a good-enough program for reaching the goal, but we have assumed an ideal frog that can jump an infinite number of times within a finite duration.

Temporal Sum

Of course, *real* motion, especially on a racetrack, is not modeled by a leaping frog. Supposing we observe a rocket moving with constant velocity V moving along a straight track of length L, one may *still* slice up the motion in a Zeno-inspired fashion. Let t_1 be the time needed to traverse half of the racetrack. Similarly, let t_2 be the time needed to cover one quarter of the track, etc. As a typical application of 'rate' problem, we have

$$\frac{L}{2} = V \cdot t_1 \qquad\qquad \frac{L}{4} = V \cdot t_2 \qquad\qquad \frac{L}{8} = V \cdot t_3 \, ,$$

and so on.

To proceed, solve for each t_i and take the sum

$$t_1 + t_2 + t_3 + \cdots = \frac{L}{V} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) \, ,$$

where the familiar parenthesized series can be replaced by one. On the left, we have an infinite sum of time intervals

$$\sum_{k=1}^{\infty} t_k = \frac{L}{V} \, ,$$

but on the right, we simply have L/V, which is surely a finite number. This result is none other than the equation capturing the total distance L, total time T, and constant velocity, $V = L/T$.

13 Factorial Operator

An interesting generalization of multiplication is *factorial* operator, appearing as an exclamation point (!) directly after an integer. The factorial operator transforms an integer into a product of the base number and all lesser integers, as in

$$4! = 4 \times 3 \times 2 \times 1 \; .$$

Factorials aren't easily added or subtracted, but division and multiplication is straightforward:

$$\frac{6!}{3!} \times 4! = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} \times 4 \times 3 \times 2 \times 1 = 4 \times 6!$$

For a general integer n, we can play similar games:

$$\begin{aligned} n! &= n\,(n-1)! \\ n! &= n\,(n-1)\,(n-2)! \\ \frac{n!}{(n-3)!} &= n\,(n-1)\,(n-2) \end{aligned}$$

13.1 Binomial Coefficients

Recall that the expansions of $(a+b)^n$ can be attained for any value of n. With enough patience, we eventually generate Pascal's triangle to keep track of the coefficients. That is,

$$\begin{aligned} (a+b)^0 &= 1 \\ (a+b)^1 &= a + b \\ (a+b)^2 &= a^2 + 2ab + b^2 \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \end{aligned}$$

leads to:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The individual entries of Pascal's triangle are called *binomial coefficients*, and are contained in the neat formula

$$C_n^m = \frac{n!}{m!\,(n-m)!} \; ,$$

where n is the row counting downward from the top, starting from $n = 0$. The number m denotes a given entry counting from the left, also starting from $m = 0$. For example, to generate the list 1, 3, 3, 1, we set $n = 3$ and write:

$$C_3^0 = \frac{3!}{0!\,(3-0)!} = 1 \qquad C_3^1 = \frac{3!}{1!\,(3-1)!} = 3$$

$$C_3^2 = \frac{3!}{2!\,(3-2)!} = 3 \qquad C_3^3 = \frac{3!}{3!\,(3-3)!} = 1$$

Using binomial coefficients, a general binomial expansion is easily written in summation notation:

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

Negative Exponents

Binomial coefficients come into play when handling expansions of $1/(a+b)^n$. Recall from our study of geometric series that the equations

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 - 10x^3 + \cdots$$

contain infinite sequences on the right. To make sense of these, write a version of Pascal's triangle generated by $(a-b)^n$ to get:

$$\begin{array}{ccccccccc} & & & & +1 & & & & \\ & & & +1 & & -1 & & & \\ & & +1 & & -2 & & +1 & & \\ & +1 & & -3 & & +3 & & -1 & \\ +1 & & -4 & & +6 & & -4 & & +1 \end{array}$$

Starting at the nth row down on the left edge, and then reading 'south-east', the above coefficients $\tilde{C}_n^m$ match the diagonal rows of the triangle. For a given diagonal n, the sequence runs through all integer m from zero to infinity. That is,

$$\frac{1}{(1+x)^n} = 1 + \tilde{C}_n^1 x + \tilde{C}_{n+1}^2 x^2 + \tilde{C}_{n+2}^3 x^3 + \cdots = \sum_{m=0}^{\infty} \tilde{C}_{n+m-1}^m x^m$$

Note that for even values of m, the coefficients $\tilde{C}_n^m$ are positive, and thereby equal to C_n^m, whereas for for negative m, the coefficients are simply $-C_n^m$. To capture this, we write

$$\frac{1}{(1+x)^n} = \sum_{m=0}^{\infty} (-1)^m \, C_{n+m-1}^m x^m \, ,$$

where the coefficient C_{n+m}^{m} resolves to

$$C_{n+m-1}^{m}=\frac{(m+n-1)!}{m!\,(n-1)!}\,.$$

For example, to generate the coefficients for $1/\left(1+x\right)^4$, we set $n=4$ and start grinding for all positive integer m:

$$C_3^0=\frac{(0+4-1)!}{0!\,(4-1)!}=1 \qquad C_4^1=\frac{(1+4-1)!}{1!\,(4-1)!}=4$$

$$C_5^2=\frac{(2+4-1)!}{2!\,(4-1)!}=10 \qquad C_6^3=\frac{(3+4-1)!}{3!\,(4-1)!}=20$$

Of course, the whole apparatus generalizes for expansions of $1/\left(a+b\right)^n$. Jumping right to the result, this is:

$$\frac{1}{\left(a+b\right)^n}=\sum_{m=0}^{\infty}\left(-1\right)^m C_{n+m-1}^{m}\frac{a^m}{b^{n+m}}$$

For another order-four example, we set $n=4$ in the above to generate the expansion of $1/\left(a+b\right)^4$:

$$\begin{aligned}\frac{1}{\left(a+b\right)^4}&=C_3^0\frac{a^0}{b^4}-C_4^1\frac{a^1}{b^5}+C_5^2\frac{a^2}{b^6}-C_6^3\frac{a^3}{b^7}+\cdots\\&=\frac{1}{b^4}-\frac{4a}{b^5}+\frac{10a^2}{b^6}-\frac{20a^3}{b^7}+\cdots\end{aligned}$$

14 Logarithm Operator

In the same sense that subtraction is the inverse of addition, and that division is the inverse of multiplication, the exponent operator can also be inverted by an operator called the *logarithm*.

14.1 Base-Ten Logarithm

Consider the perfect power-of-ten numbers

$$100 = 10^2 \qquad 1000 = 10^3 \qquad 10000 = 10^4 \,,$$

and so on. Introducing the *base-ten logarithm*, we can 'solve for' the exponent that brings the base number 10 to higher powers. In practice, this appears as

$$\log_{10}(100) = \log_{10}\left(10^2\right) = 2 \qquad \log_{10}(1000) = \log_{10}\left(10^3\right) = 3 \,,$$

where we clearly have

$$\log_{10}\left(10^N\right) = N \,.$$

In other words, the base-ten logarithm is a way of 'solving for' exponents occurring in an equation, so long as the base number is ten.

The logarithm number supports base numbers apart from 10. Supposing we start with

$$49 = 7^2 \qquad 343 = 7^3 \,,$$

we can apply a base-7 logarithm to dig out the exponents:

$$\log_7(49) = \log_7\left(7^2\right) = 2 \qquad \log_7(343) = \log_7\left(7^3\right) = 3$$

14.2 Arbitrary-Base Logarithm

The whole apparatus generalizes to any base number. If we have a quantity

$$Y = X^N \,,$$

we may apply a base-X logarithm operation as

$$\log_X(Y) = \log_X\left(X^N\right) = N \,.$$

Unit Exponent

For the special case $N = 1$, we establish an important identity:

$$\log_X\left(X^1\right) = \log_X(X) = 1$$

Inverse Logarithm

The operation that 'undoes' the logarithm is simply the exponent operator. To see this plainly, note first that $X^A = X^A$ always holds, and then use $A = \log_X \left(X^N\right) = N$ to write

$$X^{\log_X \left(X^N\right)} = X^N \ ,$$

and then substitute $Y = X^N$ to get

$$X^{\log_X (Y)} = Y \ .$$

Evidently, the combination $X^{\log_X (Y)}$ leaves the number Y unchanged.

Isolating the Base

Starting from $Y = X^N$, we easily isolate the base number X by raising each side to the power $1/N$:

$$Y^{1/N} = X^{N/N} = X$$

14.3 Change of Base

Any quantity $Y = X^N$ has an equivalent representation using a different base number q raised to a different exponent r, i.e. $Y = q^r$. To establish this, apply the base-X logarithm to both sides of X^N

$$\log_X (Y) = N \ ,$$

so we may eliminate N to write

$$Y = X^{\log_X (Y)} \ .$$

In this form, we see that X can easily be swapped with a different arbitrary number q such that

$$Y = q^{\log_q (Y)} = q^r \ ,$$

where the exponent term is

$$r = \log_q (Y) \ .$$

On the other hand, if r is given q is unknown, we simply use

$$q = Y^{1/r} \ .$$

14.4 Identities

Addition Identity

Suppose we have two logarithmic quantities $\log_X (a)$, $\log_X (b)$ of the same base number X, and we are interested in the sum

$$S = \log_X (a) + \log_X (b) \ .$$

To proceed, raise each side as an exponent with X as the base number

$$X^S = X^{\log_X(a)+\log_X(b)} = X^{\log_X(a)} \cdot X^{\log_X(b)} ,$$

where the rule $z^{c+d} = z^c z^d$ has been used. Next, note that the right side reduces to the product ab, or

$$X^S = ab .$$

Finally, apply the $\log_X$-operator to each side to find $\log_X\left(X^S\right) = S$, and arrive at

$$\log_X(ab) = \log_X(a) + \log_X(b) \ .$$

Of course, this result generalizes to more than two terms:

$$\log_X(abc\cdots) = \log_X(a) + \log_X(b) + \log_X(c) + \cdots \ .$$

As a corollary to the addition identity, we notice that multiplying the product $abc\cdots$ by a factor of one, thus not changing the product, is equivalent to adding zero to the right side, namely

$$\log_X(1) = 0 ,$$

for any base number.

Subtraction Identity

Suppose we have two logarithmic quantities $\log_X(a)$, $\log_X(b)$ of the same base number X, and we are interested in the difference

$$D = \log_X(a) - \log_X(b) \ .$$

To proceed, raise each side as an exponent with X as the base number

$$X^D = X^{\log_X(a)-\log_X(b)} = X^{\log_X(a)} \cdot \frac{1}{X^{\log_X(b)}} ,$$

where the rule $z^{c-d} = z^c/z^d$ has been used. Next, note that the right side reduces to the ratio a/b, or

$$X^D = a/b .$$

Finally, apply the $\log_X$-operator to each side to find $\log_X\left(X^D\right) = D$, and arrive at

$$\log_X\left(\frac{a}{b}\right) = \log_X(a) - \log_X(b) \ .$$

Product Identity

For a base number X, consider a logarithmic quantity $\log_X(a)$, along with an arbitrary number b, and let us take the product

$$P = b \cdot \log_X(a) \ .$$

To proceed, raise each side as an exponent with X as the base number

$$X^P = X^{b \cdot \log_x(a)} = \left(X^{\log_x(a)}\right)^b \ ,$$

where the rule $z^{cd} = (z^c)^d$ has been used. Next, note that the right side reduces to the exponent a^b, or

$$X^P = a^b \ .$$

Finally, apply the $\log_X$-operator to each side to find $\log_X\left(X^P\right) = P$, and arrive at

$$\log_X\left(a^b\right) = b \cdot \log_X(a) \ .$$

15 Scientific Notation

In *scientific notation*, any number X should be reported in the form

$$X = A \times 10^N \,,$$

where A is a number confined to $|A| < 10$, and N is an integer. A quantity X can often be quickly converted to scientific notation by multiplying or dividing by the required factor of ten, provided ten is the base number.

To proceed, let us recast the variable $A = 10^C$ or $C = \log_{10}(A)$. With this, the variable C is confined to the interval $0 < C < 1$, and the above becomes

$$X = 10^C \times 10^N = 10^{C+N} \,.$$

Apply the base-ten logarithm operator to each to side to get

$$\log_{10}(X) = \log_{10}(C+N) = C + N \,.$$

In practice, this means we take the result $\log_{10}(X)$, assigning the fractional component to C, and the integer component to N.

15.1 Integer Operator

Let us define an *integer operator* that simply 'shaves off' any decimals from a number. For instance if we start with $N = 5.74$, then $\text{int}(N) = \text{int}(5.74) = 7$. The integer operator does *no* rounding. That is, $\text{int}(3.99999) = 3$. Using the integer operator, we can solve for C and N in the above, giving:

$$N = \text{int}(\log_{10}(X))$$

$$\begin{aligned} C &= \log_{10}(X) - N \\ &= \log_{10}(X) - \text{int}(\log_{10}(X)) \end{aligned}$$

Now, we can write any number X in scientific notation:

$$\begin{aligned} X &= A \times 10^N \\ X &= 10^{\log_{10}(X) - \text{int}(\log_{10}(X))} \times 10^{\text{int}(\log_{10}(X))} \end{aligned}$$

To tighten things up, let us substitute

$$Y = \log_{10}(X) = C + N \,,$$

so we finally have

$$X = 10^{Y - \text{int}(Y)} \times 10^{\text{int}(Y)} = 10^C \times 10^N \,.$$

15.2 Factorials and Scientific Notation

Recall that the so-called 'factorial' of a number N goes like

$$N! = N\,(N-1)\,(N-2)\cdots(1) \ ,$$

with N total factors. To convert such numbers to scientific notation, apply the base-ten logarithm operator and use the addition identity to write

$$\log_{10}(N!) = \log_{10}(N) + \log_{10}(N-1) + \log_{10}(N-2) + \cdots + \log_{10}(1) \ .$$

Using summation notation, the above condenses to

$$\log_{10}(N!) = \sum_{k=0}^{N-1} \log_{10}(N-k) = Y \ ,$$

where, as previously, Y resolves to the sum of $C + N$, with N being an integer and C being the fractional component. Thus, we have

$$N! = 10^{(\log_{10}(N!))} = 10^{Y} = 10^{C} \times 10^{N} \ ,$$

which takes familiar form:

$$N! = 10^{Y-\mathrm{int}(Y)} \times 10^{\mathrm{int}(Y)}$$